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Families of orthogonal functions defined by the Weyl groups of compact Lie groups

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SOMMAIRE

Plusieurs familles de fonctions spéciales de plusieurs variables, appelées fonctions d'orbites, sont définies dans le contexte des groupes de Weyl de groupes de Lie simples compacts/d'algèbres de Lie simples. Ces fonctions sont étudiées depuis près d'un siècle en raison de leur lien avec les caractères des représentations irréductibles des algèbres de Lie simples, mais également de par leurs symétries et orthogonalités. Nous sommes principalement intéressés par la description des relations d'orthogonalité discrète et des transformations discrètes correspondantes, transformations qui permettent l'utilisation des fonctions d'orbites dans le traitement de données multidimensionnelles. Cette description est donnée pour les groupes de Weyl dont les racines ont deux longueurs différentes, en particulier pour les groupes de rang 2 dans le cas des fonctions d'orbites du type E et pour les groupes de rang 3 dans le cas de toutes les autres fonctions d'orbites.

MOT CLÉS: transformations discrètes, interpolation, fonctions d'orbites, polytopes, groupes de Weyl

SUMMARY

Several families of multivariable special functions, called orbit functions, are defined in the context of Weyl groups of compact simple Lie groups/Lie algebras. These functions have been studied for almost a century now because of their relation to characters of irreducible representations of Lie algebras, their symmetries and orthogonalities. Our main interest is the description of discrete orthogonality relations and their corresponding discrete transforms which allow the applications of orbit functions in the processing of multidimensional data. This description is provided for the Weyl group of different lengths of root, in particular groups of rank 2 for so-called $E-$orbit functions and of rank 3 for all the other families of special functions.

KEYWORDS: discrete transforms, interpolation, orbit functions, polytopes, Weyl groups

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Děkuji své mamince za trpělivost a podporu během celého mého studia.

Chapter 1

INTRODUCTION

In this work we study infinite families of functions which are related to the Weyl groups of semisimple Lie algebras/Lie groups. These so-called **orbit functions** are complex functions depending on n real variables, where n is the rank of the underlying Lie algebra. Since they have pertinent mathematical properties and extensive use in physics they can be considered as special functions [**52**].

The first two families of orbit functions, $C-$ and $S-$functions were introduced in [**41**] as the contribution to the character of an irreducible representation from a single Weyl group orbit. In particular, every character can be written as a linear combination of $C-$functions, the ratio of $S-$ functions appears in the Weyl character formula. Working with orbit functions instead of characters brought a significant simplification of the computation in many problems [**40, 41**]. In the case of the Weyl group of A_1, the $C-$ and $S-$functions become, up to a constant, the common cosine and sine functions. As a generalization of the exponential function, the family of $E-$functions was introduced in [**51**].

By definition, $C-$ and $S-$functions are (skew-)symmetric with respect to the Weyl group and $E-$functions are symmetric with respect to the even Weyl subgroup. Continuous orthogonality of the functions when integrated over a finite region F, which is the fundamental domain of the affine Weyl group W^{aff}, is a corollary of the orthogonality of characters [**60**]. It is summarized along with the discrete orthogonality in [**43**]. It is also well known [**52**] that orbit functions are eigenfunctions of the Laplace operator. All these and other pertinent properties of orbit functions were reviewed and further developed in a series of papers: [**26**] for $C-$functions, [**24**] for $S-$functions and [**25**] for $E-$functions.

In [**49**] the authors studied the correspondence between symmetric and antisymmetric generalizations of sine and cosine functions and the orbit functions of C_2. Two new families, so-called S^s- and S^l-functions, were defined. They were described for the Weyl group of G_2 in [**62**] and the general definition for all the Weyl groups with different lengths of roots was introduced in [**37**]. The main tool for the definition are the sign homomorphisms $\sigma : W \to \{\pm 1\}$, which are described in Section 3.3.1. The S^s- and S^l-functions have analogous properties as the $C-$ and $S-$functions, namely the continuous and discrete orthogonality. They differ by the behavior on the boundary of the fundamental domain.

The notations of the two new families of functions led to a natural generalization of the $E-$functions. The relations among the original three families is given schematically by $E = C + S$. Using five other combinations of orbit functions gives five new families of orbit functions. The precise definition of all six families of $E-$functions uses the concept of the sign homomorphisms and three normal subgroups of the Weyl group W, see Section 3.3.2. The functions are described in detail for the Weyl groups of C_2 and G_2 in the paper [**2**] which is the content of Chapter 3 of this thesis.

Our main interest lies in the discretization of orbit functions and their applications in n-dimensional data processing. The basic method consists of setting up a lattice fragment F_M in F, where M is an integer of our choice determining the density of the lattice, then sampling the orbit functions at the points of F_M and showing the orthogonality relations of orbit functions with respect to a scalar product defined using the points of F_M. Using this we can perform a Fourier-like transform of any function f defined on the points of F_M and, in consequence, by replacing the lattice variable by a continuous one, we obtain an interpolation of f. This method is advantageous especially as it can be used in any dimension, unlike a lot of other interpolating methods, which are often restricted to low dimensions. Necessary details for this method are described in several papers [**16, 17, 18, 19**].

The work is organized as follows. The rest of this chapter contains a review of some terms which are necessary for the text. The Chapters 2–4 consist of the following papers [**1, 2, 3**]:

(1) **The rings of n-dimensional polytopes**

In this article we study in detail orbits of the action of a Coxeter group G on the Euclidean space $\mathbb{R}^n$. We view the points of each orbit as vertices of a n-dimensional G-invariant polytope centered at the origin, so-called G-polytopes. In this work we do not distinguish crystallographic and non-crystallographic Coxeter groups.

In Section 2.4 we present an algorithm for computing points of the orbits starting from the dominant point (the point belonging to the fundamental domain of the corresponding Coxeter group) and give several examples for the groups of rank 2 and 3.

Section 2.5 reviews a method for describing G-polytopes in any dimension. Using the so-called diagram decorations we compute the number of faces of all dimensions as well as other characteristics of the polytopes.

In Section 2.6 we present the multiplication of G-polytopes and its decomposition. Section 2.7 studies the decomposition of symmetrized powers of orbits. Finally, we describe several invariants of G-polytopes.

The main contribution of this article is the unification of the study of Weyl groups and non-crystallographic Coxeter groups. In the case of Weyl groups, the detail description of orbits is essential for further study of orbit functions.

My contribution to the article lies especially in writing a part of the text and computing particular cases in the sections 2.4, 2.5, 2.6 and 2.8, i.e., examples of orbits, orbit product decomposition and indices.

(2) **Six types of $E-$functions of the Lie groups $O(5)$ and $G(2)$**

This article brings to the literature new families of orbit functions which are a generalization of E-orbit functions [**25**]. The five new families are defined for the Weyl group of all simple Lie groups with two root lengths. With a small modification, all the basic properties of orbit functions are satisfied, namely the (anti-)invariance with respect to a certain subgroup of the Weyl group, as well as continuous and discrete orthogonality.

In order to provide all the details, our consideration is limited to the description of E-functions of the Weyl group of rank 2, i.e., the groups of C_2 and G_2.

After introducing the notion of sign homomorphisms and pertinent subgroups of the corresponding Weyl group we define the even orbit functions in Section 3.4 and mixed orbit functions in Section 3.5. For each family of functions we write explicitly its symmetry, fundamental domain, continuous and discrete orthogonality and the discrete transform. We give several examples. This article provides in detail all the information necessary for the applications.

My contribution to the article, beside writing a part of the text, is the following: definition of the $E-$orbit functions using the subgroups of the Weyl group W, description of all kinds of orbits of C_2 and G_2, giving the explicit formulas of the grids F_M, Λ_M for all types of $E-$orbit functions and the content of the Section 3.6.

(3) **Four families of Weyl group orbit functions of B_3 and of C_3**

In this article we follow a similar approach as in the previous one. Our main aim is to study families of orbit functions $C-$, $S-$, S^s- and S^l- for the Weyl groups with two root lengths of rank 3, i.e., the Weyl groups of B_3 and C_3. We provide a detailed description of the fundamental domains and the corresponding lattice fragments in Section 4.3. We emphasize the discrete orthogonality of S^s- and S^l- functions and the discrete expansion in Section 4.4. We present an interpolation method of a function f sampled on the lattice fragment. We conclude with two examples using S^s- and S^l-functions interpolations.

My contribution to the article lies in writing the major part of the text, providing all the tables and figures, giving the explicit formulas of the grids F_M, Λ_M and computing the examples of interpolation.

The next five sections contain a review of basic definitions and theorems. Some of the terms are introduced in the next chapters for specific cases, e.g., for the groups of rank 2 and 3 only. Most of the sections concerning root systems and reflection groups is taken from [**20, 21**], the part concerning the types of bases of $\mathbb{R}^n$ and the affine Weyl groups is taken from [**18, 26**].

1.1. Root systems

Let $\mathbb{R}^n$ be the real n-dimensional Euclidean space with scalar product denoted by $\langle\cdot,\cdot\rangle$. Let $\alpha \in \mathbb{R}^n$ be a non-zero vector. The **reflection** fixing the hyperplane X_α passing through the origin and orthogonal to α is denoted r_α. It is given by the explicit formula

$$r_\alpha x = x - \frac{2\langle x,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha \tag{1.1}$$

for any $x \in \mathbb{R}^n$.

We can see immediately some basic properties of a reflection. For every non-zero vector $\alpha \in \mathbb{R}^n$ we have

- $r_\alpha 0 = 0$,
- $r_{c\alpha} = r_\alpha$ for every $c \neq 0$,
- r_α is orthogonal, i.e., $\langle x,y\rangle = \langle r_\alpha x, r_\alpha y\rangle$ for every $x,y \in \mathbb{R}^n$.

Definition 1.1. *Let $\Phi = \{\alpha_1,\dots,\alpha_k\}$ be a finite set of non-zero vectors in $\mathbb{R}^n$. The set Φ is called a **root system** of **rank** n if it satisfies the following axioms:*

(R1) Φ *spans* $\mathbb{R}^n$,

(R2) *if* $\alpha \in \Phi$ *then the only other multiple of* α *in* Φ *is* $-\alpha$,

(R3) Φ *is invariant with respect to* r_{α_i} *for every* $i \in \{1,\dots,k\}$.

Elements of a root system are called **roots**. Beside roots we define **coroots** by

$$\alpha_i^\vee := \frac{2\alpha_i}{\langle\alpha_i,\alpha_i\rangle}\,.$$

The set of coroots $\Phi^\vee = \{\alpha_1^\vee,\dots,\alpha_k^\vee\}$ is also a root system in $\mathbb{R}^n$ called a **dual root system**.

Some literature demands one more axiom to be satisfied, *(R4)* $\dfrac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \in \mathbb{Z}$ for every $i, j \in \{1,\ldots,k\}$.

We call a root system fulfilling the axioms *(R1–4)* a **crystallographic** root system. If the axiom *(R4)* is omitted, we speak of non-crystallographic root system.

We define the following subset of a root system:

Definition 1.2. *A **base** of a root system Φ is a subset Δ which is a basis of $\mathbb{R}^n$ and has the following property: each root $\beta \in \Phi$ can be written as a linear combination of roots from Δ,*

$$\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha\,,$$

such that the coefficients k_α are all nonnegative or all non positive integers.

In Chapter 3 the notation $\Phi(\alpha)$ will be used for the set of simple roots.

The elements of a base are called **simple** roots. The number $\mathrm{ht}(\beta) = \sum_{\alpha\in\Delta} k_\alpha$ is called the **height** of the root β relative to the base Δ. The fact that every root system has a base is not trivial and the proof can be found for example in [**20**, p. 48]. The base of a root system is not unique, we fix one in the correspondence with Dynkin diagrams (defined below), in the following we will speak of the α-basis of $\mathbb{R}^n$. If $\mathrm{ht}(\beta) > 0$, we call β a **positive root**, otherwise a **negative root**. Every base gives us a partial order $\prec$ of the roots defined by $\beta \prec \beta'$ if and only if $\beta' - \beta$ is a sum of positive roots or $\beta = \beta'$.

A root system is called **irreducible** if it cannot be written as a union of two proper subsets such that each root from the first set is orthogonal to each root from the second set (or equivalently if the same holds for any base of Φ). Irreducible root systems have some special properties, the following lemma is proved in [**20**, p. 52, lemma A,C].

Lemma 1.1. *Let Φ be a crystallographic and irreducible root system in $\mathbb{R}^n$. Then*

(a) at most two different lengths of roots occur in Φ,

(b) there is a unique highest root ξ with respect to the partial order $\prec$ corresponding to a fixed base Δ.

In the case of two different root lengths we speak of **long** and **short** roots. If there is just one root length, we call the roots long. As the definition of the root system considers only ratios of the roots lengths and not the lengths themselves, we use the convention $\langle \alpha, \alpha \rangle = 2$ for α long. In the non-crystallographic case we can consider all roots to have the same length and use the same convention $\langle \alpha, \alpha \rangle = 2$. The highest root is always long.

The space $\mathbb{R}^n$ is divided by the hyperplanes X_α, where $\alpha \in \Phi$, into several components. They are called **Weyl chambers**. The **fundamental** (or **dominant**) Weyl chamber is the one composed of $x \in \mathbb{R}^n$ such that $\langle x, \alpha \rangle > 0$ for every $\alpha \in \Delta$. The closure of the fundamental Weyl chamber is denoted by D_+. The fundamental Weyl chamber depends on the choice of Δ, in fact Weyl chambers and bases are in one-to-one correspondence.

1.2. Reflection groups

Definition 1.3. *Let Φ be a root system in $\mathbb{R}^n$ with a base $\Delta = \{\alpha_1, \dots, \alpha_n\}$. A* ***reflection group*** *G is a subgroup of $GL_n(\mathbb{R})$ generated by reflections $r_1, \dots, r_n$, where $r_i := r_{\alpha_i}$.*

In the case of crystallographic root system reflection groups are called crystallographic or **Weyl groups**, as they are isomorphic to the Weyl groups of semisimple Lie algebras. In general, reflection groups are finite **Coxeter groups** [**21**], i.e., they can be defined as a group generated by a (finite) set of generators $r_1, \dots, r_n$ which satisfy

$$r_k^2 = 1, \quad (r_i r_j)^{m_{ij}} = 1 \quad i, j, k \in \{1, \dots, n\}\,. \tag{1.2}$$

for some integers $m_{ij} = m_{ji} \geq 2$ if $i \neq j$. When we identify the generators with reflections corresponding to simple roots $\alpha_1, \dots, \alpha_n$, the angle between two roots α_i and α_j is $\pi - \frac{\pi}{m_{ij}}$.

The orbit of $\lambda \in \mathbb{R}^n$ under the action of G is denoted by $G\lambda$ or $G(\lambda)$, i.e., $G\lambda = \{w\lambda \,|\, w \in G\}$.

The reflection group G corresponding to a root system Φ with a base Δ has the following properties [**20**, p. 51]:

Lemma 1.2. *(1) G permutes the Weyl chambers: if $x \in \mathbb{R}^n \setminus \sum_{\alpha \in \Phi} X_\alpha$ there exists $w \in G$ such that $w(x)$ lies in the fundamental Weyl chamber.*

(2) G permutes the bases: if Δ' is another base, there exists $w \in G$ such that $w(\Delta') = \Delta$, moreover if $w(\Delta) = \Delta$, then w is the identity of G.

(3) If Φ is irreducible then the orbit of any $x \in \mathbb{R}^n$ spans $\mathbb{R}^n$.

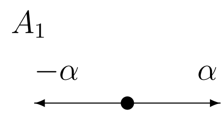

FIG. 1.1. Root system of rank 1

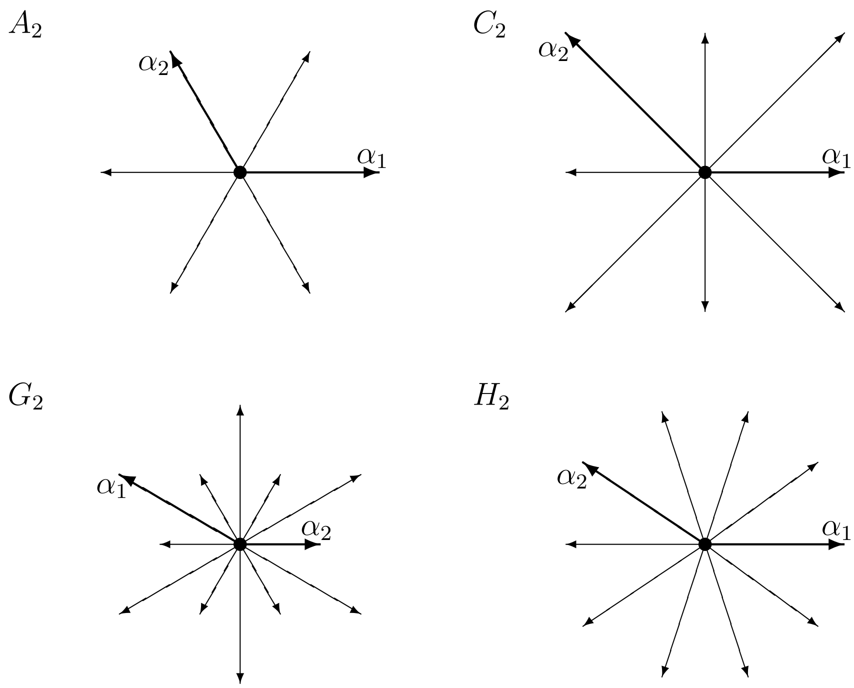

FIG. 1.2. Exemples of irreducible root systems of rank 2

For $n = 1$ there is only one root system (Figure 1.1), the corresponding reflection group contains only the identity and one reflection r. It is denoted A_1.

For rank two there are infinitely many irreducible root systems, three of them are crystallographic. The Figure 1.2 shows the crystallographic systems and one of the non-crystallographic ones. The corresponding reflection groups are isomorphic to dihedral groups. We denote the simple roots α_1 and α_2 and let $m := m_{12}$, hence $(r_1 r_2)^m = 1$. The angle between the simple roots is $\frac{(m-1)\pi}{m}$.

For $m = 2$ the root system is the only reducible case and its reflection group can be written as $A_1 \times A_1$. There are three crystallographic cases, the Weyl groups A_2, C_2 and G_2, and infinitely many non-crystallographic groups denoted $H_2(m)$ (some authors **[21]** use the notation $I_2(m)$). By convention we write only H_2 in the case when $m = 5$.

The complete classification of reflection groups corresponding to irreducible root systems has been done, for Weyl groups in [**20**] and for non-crystallographic groups in [**21**]. For the crystallographic groups we have four infinite families $A_n, n \geq 1$, $B_n, n \geq 3$, $C_n, n \geq 2$ and $D_n, n \geq 4$ and five exceptional groups E_6, E_7, E_8, F_4 and G_2. There are also two more non-crystallographic groups, H_3 and H_4.

For the reflection groups of rank greater than two we use other ways to describe them, mainly Cartan matrix and Dynkin diagram. It is shown [**20**] that both Dynkin diagram and Cartan matrix determine the root system (therefore the reflection group) up to isomorphism, where an isomorphism of a root system is $\phi \in GL_n(\mathbb{R})$ such that $\langle \phi(\alpha), \phi(\beta) \rangle = \phi(\langle \alpha, \beta \rangle)$ for each $\alpha, \beta \in \Phi$. A detailed description of Dynkin diagrams is found in Section 2.2. The **Cartan matrix** $C(G)$ corresponding to the group G generated by $\{r_1, \ldots, r_n\}$ is defined as

$$(C(G))_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}.$$

In the case of Weyl groups it is an integer matrix. The matrix depends on the ordering of simple roots but not on the choice of the base (we use the standard convention of ordering of the roots corresponding to Dynkin diagrams). Here are some examples of Cartan matrices (the rest can be found for example in [**8**]):

$$C(A_n) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}, \quad C(B_n) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -2 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix},$$

$$C(C_n) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -2 & 2 \end{pmatrix}, \quad C(D_n) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix},$$

$$C(G_2) = \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}, \quad C(H_2) = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}, \quad C(H_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}.$$

Here τ is the golden ratio, i.e., the positive root of $x^2 = x + 1$.

1.3. Bases of $\mathbb{R}^n$

Let G be a reflection group acting on $\mathbb{R}^n$ and let $\{e_1, \dots, e_n\}$ be the standard orthogonal basis of $\mathbb{R}^n$. When working with orbits of reflection groups, it is useful to work with bases which are not orthogonal: the basis of simple roots (α-basis) and the weight basis (ω-basis). The ω-basis is defined by the relation

$$\frac{2\,\langle \alpha_i, \omega_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}\,, \quad i, j \in \{1, \dots, n\}\,.$$

We can rewrite these relations using the Cartan matrix

$$\alpha_j = \sum_{k=1}^{n} (C(G))_{jk}\, \omega_k\,, \quad \omega_j = \sum_{k=1}^{n} \left(C(G)^{-1}\right)_{jk}\, \alpha_k\,.$$

In the same way as coroots we can define coweights by

$$\omega_i^\vee := \frac{2\omega_i}{\langle \alpha_i, \alpha_i \rangle}.$$

We have several types of lattices related to these basis vectors, namely

- the **root lattice**: $Q := \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n\,,$
- the **coroot lattice**: $Q^\vee := \mathbb{Z}\alpha_1^\vee + \dots + \mathbb{Z}\alpha_n^\vee\,,$
- the **weight lattice**: $P := \mathbb{Z}\omega_1 + \dots + \mathbb{Z}\omega_n\,,$
- the **coweight lattice**: $P^\vee := \mathbb{Z}\omega_1^\vee + \dots + \mathbb{Z}\omega_n^\vee\,,$
- the **cone of dominant weights**: $P^+ := \mathbb{Z}^{\geq 0}\omega_1 + \dots + \mathbb{Z}^{\geq 0}\omega_n\,.$

In general, $Q \subseteq P$, with $Q = P$ only for the groups E_8, F_4, and G_2. Similarly, $Q^\vee \subseteq P^\vee$. Moreover, $P^+ = P \cap D_+$. Other subsets of these lattices are defined when needed (in Chapters 3 and 4).

1.4. Affine Weyl groups

In this subsection we consider only crystallographic reflection groups, i.e., Weyl groups. Let Φ be a crystallographic and irreducible root system in $\mathbb{R}^n$ with the set of simple roots $\alpha_1, \dots, \alpha_n$, and let W be the corresponding Weyl group. The highest root ξ of Φ can be written in a form

$$\xi = a_1\omega_1 + \dots + a_n\omega_n = m_1\alpha_1 + \dots + m_n\alpha_n.$$

W	$a_1,\dots,a_n$	$m_1,\dots,m_n$	$m_1^\vee,\dots,m_n^\vee$
A_n	$1,0,\dots,0,1$	$1,1,\dots,1,1$	$1,1,\dots,1,1$
B_n	$0,1,\dots,0,0$	$1,2,\dots,2,2$	$2,2,\dots,2,1$
C_n	$2,0,\dots,0,0$	$2,2,\dots,2,1$	$1,2,\dots,2,2$
D_n	$0,1,\dots,0,0$	$1,2,\dots,2,1,1$	$1,2,\dots,2,1,1$
E_6	$0,0,\dots,0,1$	$1,2,3,2,1,2$	$1,2,3,2,1,2$
E_7	$1,0,\dots,0,0$	$2,3,4,3,2,1,2$	$2,3,4,3,2,1,2$
E_8	$1,0,\dots,0,0$	$2,3,4,5,6,4,2,3$	$2,3,4,5,6,4,2,3$
F_4	$1,0,0,1$	$2,3,4,2$	$2,4,3,2$
G_2	$1,0$	$2,3$	$3,2$

TAB. 1.1. Marks and dual marks

The highest root of the dual root system $\Phi^\vee$ is denoted by

$$\eta = m_1^\vee\alpha_1^\vee + \cdots + m_n^\vee\alpha_n^\vee.$$

The numbers m_i and $m_i^\vee$ are called **marks** and **dual marks** and they are listed in Table 1.1 (from [**18**]). We denote by r_ξ the reflection with respect to the hyperplane X_ξ orthogonal to ξ,

$$r_\xi x = x - \frac{2\langle x,\xi\rangle}{\langle \xi,\xi\rangle}\xi\,,$$

and by r_0 the affine reflection defined as

$$r_0 x := r_\xi x + \frac{2\xi}{\langle \xi,\,\xi\rangle}\,.$$

The **affine Weyl group** is the group of transformations generated by the reflections r_0 and $r_1,\dots,r_n$. We denote it by W^{aff}. Equivalently, we can write W^{aff} as a semidirect product of translations by vectors of $Q^\vee$ and elements of W (details can be found in [**26**]).

We are interested in describing a fundamental domain of the affine Weyl group, i.e., a set F that contains exactly one point of every W^{aff}-orbit. More precisely,

- for every $x \in \mathbb{R}^n$, there are $x' \in F$, $w \in W$ and $q^\vee \in Q^\vee$ such that $x = wx' + q^\vee$,

- if $x, x' \in F$ and $x = wx'$ for a $w \in W^{\mathrm{aff}}$, then $x = x'$.

Clearly we can consider $F \subset D_+$. In [**18**] it is proven that the simplex with vertices 0 and $\frac{\omega_i^\vee}{m_i}$ for $i \in \{1, \ldots, n\}$ is a fundamental domain of W^{aff}.

1.5. Orbit functions

Three families of orbit functions can be defined for simple Lie algebras with one root length and ten families for the algebras with two root lengths. Here we summarize the definitions of all ten families. All orbit functions are, in general, functions from $\mathbb{R}^n$ to $\mathbb{C}$ and they are indexed by a parameter $\lambda \in \mathbb{R}^n$. Nevertheless, for each family we restrict the domain to a subset of the fundamental domain of the Weyl group and the choice of λ to a subset of the lattice P. Here we skip the details since all the families are described in Chapters 3 and 4. The term orbit functions refers to the fact that the functions can be defined as summations over a single Weyl orbit, see [**26**] and [**24**] for alternative definitions. Here we use the summations over the Weyl group/subgroup.

- **C-functions**

  The C-functions, also called the symmetric orbit functions, are defined by

$$\Phi_\lambda(x) = \sum_{w \in W} e^{2\pi i \langle w\lambda,\, x\rangle} \,.$$

- **S-functions**

  The S-functions, also called the antisymmetric orbit functions, are defined by

$$\varphi_\lambda(x) = \sum_{w \in W} \det(w) e^{2\pi i \langle w\lambda,\, x\rangle} \,,$$

  where $\det(w)$ denotes determinant of w.

- **S^s- and S^l-functions**

  These functions become C- and S- functions respectively in the case of the Lie algebras with only one root length. For other algebras they are defined as

$$\varphi^s_\lambda(x) = \sum_{w\in W} \sigma^s(w)\, e^{2\pi i \langle w\lambda,\, x\rangle}\,,$$
$$\varphi^l_\lambda(x) = \sum_{w\in W} \sigma^l(w)\, e^{2\pi i \langle w\lambda,\, x\rangle}\,,$$

where the homomorphisms σ^s and σ^l are defined in Section 3.3.1.

- **Even orbit functions**

  There are 6 families of E-orbit functions which can be split into two groups, even orbit functions and mixed even orbit functions. Only the first family is defined for all Weyl groups. The sums in the definition are over subgroups of W which are noted W^e, W^s and W^l and are defined in Section 3.3.2.

  Even orbit functions are defined by

$$\Xi_\lambda^{e+}(x) = \sum_{w \in W^e} e^{2\pi i \langle w\lambda,\, x\rangle} = \frac{1}{2}\left(\Phi_\lambda(x) + \varphi_\lambda(x)\right),$$

$$\Xi_\lambda^{s+}(x) = \sum_{w \in W^s} e^{2\pi i \langle w\lambda,\, x\rangle} = \frac{1}{2}\left(\Phi_\lambda(x) + \varphi_\lambda^s(x)\right),$$

$$\Xi_\lambda^{l+}(x) = \sum_{w \in W^l} e^{2\pi i \langle w\lambda,\, x\rangle} = \frac{1}{2}\left(\Phi_\lambda(x) + \varphi_\lambda^l(x)\right).$$

- **Mixed even orbit functions**

  Mixed even orbit functions are defined by

$$\Xi_\lambda^{e-}(x) = \sum_{w \in W^e} \sigma^s(w) e^{2\pi i \langle w\lambda,\, x\rangle} = \frac{1}{2}\left(\varphi_\lambda^s(x) + \varphi_\lambda^l(x)\right),$$

$$\Xi_\lambda^{s-}(x) = \sum_{w \in W^s} \sigma^l(w) e^{2\pi i \langle w\lambda,\, x\rangle} = \frac{1}{2}\left(\varphi_\lambda(x) + \varphi_\lambda^s(x)\right),$$

$$\Xi_\lambda^{l-}(x) = \sum_{w \in W^l} \sigma^s(w) e^{2\pi i \langle w\lambda,\, x\rangle} = \frac{1}{2}\left(\varphi_\lambda(x) + \varphi_\lambda^l(x)\right).$$

Chapter 2

THE RINGS OF N-DIMENSIONAL POLYTOPES

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Abstract: Points of an orbit of a finite Coxeter group G, generated by n reflections starting from a single seed point, are considered as vertices of a polytope (G-polytope) centered at the origin of a real n-dimensional Euclidean space. A general efficient method is recalled for the geometric description of G-polytopes, their faces of all dimensions and their adjacencies. Products and symmetrized powers of G-polytopes are introduced and their decomposition into the sums of G-polytopes is described. Several invariants of G-polytopes are found, namely the analogs of Dynkin indices of degrees 2 and 4, anomaly numbers, and congruence classes of the polytopes. The definitions apply to crystallographic and non-crystallographic Coxeter groups. Examples and applications are shown.

2.1. Introduction

Finite groups generated by reflections in a real Euclidean space $\mathbb{R}^n$ of n dimensions, also called finite Coxeter groups, are split into two classes: crystallographic and non-crystallographic groups [**21, 22**]. The crystallographic groups are the Weyl groups of compact semisimple Lie groups. They are an efficient tool for uniform description of the semisimple Lie groups/algebras [**7, 66, 20**], and they have proven to be an indispensable tool in extensive computations with the representations of such Lie groups or Lie algebras (see for example [**13**] and references therein).

Underlying such applications are two facts: (i) Most of the computation can be performed in integers by working with the weight systems of the representations involved in a problem, and (ii) the weight system of a representation of a compact semisimple Lie group/Lie algebra consists of several Weyl group orbits of the weights, many of them occurring more than once. Practical importance of the orbits apparently emerged only in [**40, 41**], where truly large scale computations were anticipated.

The crystallographic Coxeter groups are called Weyl groups and denoted by W. Any finite Coxeter group, crystallographic or not, is denoted by G. A difference between the two cases which is of practical importance to us, is that, lattices with W-symmetries are common crystallographic lattices, while lattices of non-crystallographic types are dense everywhere in $\mathbb{R}^n$.

Non-crystallographic finite Coxeter groups are of extensive use in modeling aperiodic point sets with long-range order ('quasicrystals') [**44, 36, 10**]. Outside traditional mathematics and mathematical physics, a new line of application of Coxeter group orbits can be found in [**65**]; see also the references therein.

Additional applications of Weyl group orbits are found in [**12, 63, 64, 5, 4**]. Both crystallographic and non-crystallographic Coxeter groups can be used for building families of orthogonal polynomials of many variables [**11**].

In recent years, another field of applications of W-orbits is emerging in harmonic analysis. Multidimensional Fourier-like transforms were introduced and are currently being explored in [**51, 43, 26, 24**], where W-orbits are used to define families of special functions, called orbit functions [**51**], which serve as the kernels of the transforms. They differ from the traditional special functions [**29**]. The number of variables, on which the new functions depend, is equal to the rank of a compact semisimple Lie group that provides the Weyl group. Two properties of the transforms stand out: Such special functions are orthogonal when integrated over a finite region F, and they are also orthogonal when summed up over lattice points $F_M \subset F$. The lattices can be of any density, their symmetries are prescribed by the Lie groups. Application of the non-crystallographic groups in Fourier analysis is at its very beginning [**38**].

In this paper we have no compelling reason to distinguish crystallographic and non-crystallographic reflection groups of finite order. Hence, we consider all finite Coxeter groups although from the infinitely many finite Coxeter groups in 2 dimensions (symmetry groups of regular polygons), we usually consider only the lowest few.

An orbit $G(\lambda)$ of a Coxeter group G is the set of points in $\mathbb{R}^n$ generated by G from a single seed point $\lambda \in \mathbb{R}^n$. G-orbits are not common objects in the literature, nor is their multiplication, which can be viewed in parallel to the multiplication of G-invariant polynomials $P(\lambda; x)$ introduced in subsection 6.1 (for more about the polynomials see [**11**] and the references therein).[1] Indeed, the set of exponents of all the monomials in $P(\lambda; x)$ is the set of points of the orbit $G(\lambda)$.

In this paper, we have adopted a point of view according to which the orbits $G(\lambda)$, being simpler than the polynomials $P(\lambda; x)$ or the weight systems of representations, are the primary objects of study.

The relation between the orbits of W and the weight systems of finite dimensional irreducible representations of semisimple Lie groups/algebras over $\mathbb{C}$, can be understood as follows. The character of a particular representation involves summation over the weight system of the representation, i.e. over several W-orbits. As for which orbits appear in a particular representation, this is a well known question about multiplicities of dominant weights. There is a laborious but rather fast computer algorithm for calculating the multiplicities. Extensive tables of multiplicities can be found in [**8**]; see also the references therein. Thus one is justified in assuming that the relation between a representation and a particular W-orbit is known in all cases of interest.

[1]Polynomials in 6.1 are the simplest W-invariant ones. We are not concerned about any other of their properties.

Numerical characteristics, such as congruence classes, indices of various degrees, and anomaly numbers, introduced here for W-orbits, mirror similar properties of weight systems from representation theory, which are often used in applications (for example [**12, 63, 64, 35, 61**]).

In this paper, we introduce operations on W-orbits that are well known for weight systems of representations: (i) The product of W-orbits (of the same group) and its decomposition into the sum of W-orbits; (ii) The decomposition of the k-th power of a W-orbit symmetrized by the group of permutations of k elements. New is the introduction of such operations for the orbits of non-crystallographic Coxeter groups. We intend to describe reductions of G-orbits to orbits of a subgroup $G' \subset G$ in a separate paper [**30**]. Again, the involvement of non-crystallographic groups makes the reduction problem rather unusual. Corresponding applications deserve to be explored.

The decomposition of products of orbits of Coxeter groups, as introduced here, is the core of other decomposition problems in mathematics, such as the decomposition of direct products of representations of semisimple Lie groups, the decomposition of products of certain special functions [**51**] and the decomposition of products of G-invariant polynomials of several variables [**11**]. The last two problems are completely solved in terms of orbit decompositions. The first problem requires that the multiplicities of dominant weights in weight systems of representations [**8**] be known.

We view the G-orbits from a perspective uncommon in the literature. Namely, the points of a G-orbit are taken to be vertices of an n-dimensional G-invariant polytope centered at origin, n being the number of elementary reflections generating G (at the same time it is the rank of the corresponding semisimple Lie group). The multiplication of two such polytopes/orbits, say P_1 and P_2, is the set of points/vertices obtained by adding to every point of P_1 every point of P_2. The resulting set of points is again G-invariant and thus it is a union (we say 'sum') of several G-orbits (we say 'G-polytopes'). Thus we have a ring of G-polytopes with positive integer coefficients. We recall and illustrate a general method of description of n-dimensional reflection-generated polytopes [**45, 9**].

The core of our geometric interpretation of orbits as polytopes is in the paragraph following equation (2.11). A product of orbits is a union of concentric orbits. Geometrically this can be seen as an 'onion'-like layered structure of orbits of different radii. Unlike in representation theory, where orbit points are always points of the corresponding weight lattices, in our case the seed point of an orbit can be anywhere in $\mathbb{R}^n$. In particular, a suitable choice of the seed points of the orbits, which are being multiplied, can bring some of the layers of the 'onion' structure as close or as far apart as desired. Two examples are given in the last section (see (2.30) and (2.31)).

2.2. Reflections generating finite Coxeter groups

Let α and x be vectors in $\mathbb{R}^n$. We denote by r_α the reflection in the $(n-1)$-dimensional 'mirror' orthogonal to α and passing through the origin. For any $x \in \mathbb{R}^n$, we have

$$r_\alpha x = x - \frac{2\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \,. \tag{2.1}$$

Here $\langle a, b\rangle$ denotes scalar product in $\mathbb{R}^n$. In particular, we have $r_\alpha 0 = 0$ and $r_\alpha \alpha = -\alpha$ so that $r_\alpha^2 = 1$.

A Coxeter group G is by definition generated by several reflections in mirrors that have the origin as their common point. Various Coxeter groups are thus specified by the set $\Pi(\alpha)$ of vectors α, orthogonal to the mirrors and called the simple roots of G. Consequently, G is given once the relative angles between elements of $\Pi(\alpha)$ are given.

A standard presentation of G, generated by n reflections, amounts to the following relations

$$r_k^2 = 1\,, \qquad (r_i r_j)^{m_{ij}} = 1\,, \qquad k, i, j \in \{1, \dots, n\}\,, \tag{2.2}$$

where we have simplified the notation by setting $r_{\alpha_k} = r_k$, and where m_{ij} are the lowest possible positive integers satisfying (2.2). The matrix (m_{ij}) specifies the group. The angles between the mirrors of reflections r_i and r_j are determined from the values of the exponents m_{ij}. Indeed, for $m_{ij} = p$, the angle is π/p, while the angle between α_i and α_j is $\pi - \pi/p$.

The classification of finite reflection (Coxeter) groups was accomplished in the first half of the 20th century.

2.2.1. n=1

There is just one group of order 2. Its two elements are 1 and r. We denote this group by A_1. Acting on a point a of the real line, the group A_1 generates its orbit of two points, a and $ra = -a$, except if $a = 0$. Then the orbit consists of just one point, namely the origin.

2.2.2. n=2

There are infinitely many Coxeter groups in $\mathbb{R}^2$, one for each $m_{12} = 2, 3, 4, \ldots$. Their orders are $2m_{12}$. In physics literature, these are the dihedral groups.

Note that for $m_{12} = 2$, the group is a product of two groups from $n = 1$. The reflection mirrors are orthogonal.

Our notation for the lowest five groups, generated by two reflections, and their orders, is as follows:

$$
\begin{array}{lcll}
m_{12} = 2 & : & A_1 \times A_1\,, & 4 \\
m_{12} = 3 & : & A_2\,, & 6 \\
m_{12} = 4 & : & C_2\,, & 8 \\
m_{12} = 5 & : & H_2\,, & 10 \\
m_{12} = 6 & : & G_2\,, & 12\,.
\end{array}
$$

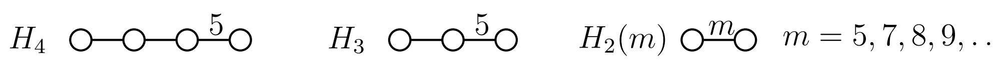
2.2.3. General case: Coxeter and Dynkin diagrams

A convenient general way to provide a specific set $\Pi(\alpha)$ is to draw a graph where vertices are traditionally shown as small circles, one for each $\alpha \in \Pi$, and where edges indicate absence of orthogonality between two vertices linked by an edge.

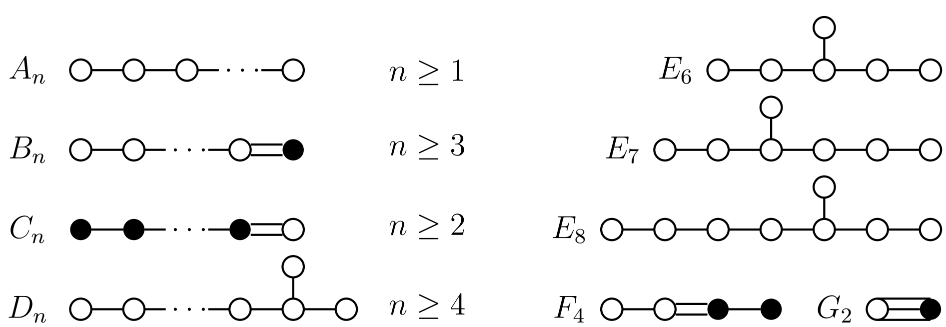
A diagram consisting of several disconnected components means that the group is a product of several pairwise commuting subgroups. Thus it is often sufficient to consider only the groups with connected diagrams.

In this paper, a Coxeter diagram is a graph providing only relative angles between simple roots while ignoring their lengths. This is done by writing m_{ij} over the edges of the diagram. By convention, the most frequently occurring value, $m_{ij} = 3$, is not shown in the diagrams. When $m_{ij} = 2$, the edge is not drawn, i.e. the nodes numbered i and j are not directly connected.

Consider the examples of Coxeter diagrams of all finite non-crystallographic Coxeter groups with connected diagrams. Note that we simply write H_2 when $m = 5$.



A Dynkin diagram is a graph providing, in addition to the relative angles, the relative lengths of the vectors from $\Pi(\alpha)$. Dynkin diagrams are used for the crystallographic Coxeter groups, frequently called the Weyl groups. There are four infinite series of classical groups and five isolated cases of exceptional simple Lie groups. Here is a complete list of Dynkin diagrams of such groups (with connected diagrams):



The names of the groups, as is traditional in Lie theory, are shown on the left of each diagram. Open (black) circles indicate longer (shorter) roots. The ratio of their square lengths is $\langle \alpha_l, \alpha_l \rangle : \langle \alpha_s, \alpha_s \rangle = 2 : 1$ in all cases except for G_2 where the ratio is $3 : 1$. Moreover, we adopt the usual convention that $\langle \alpha_l, \alpha_l \rangle = 2$. A single, double, and triple line indicates respectively the angle $2\pi/3$, $3\pi/4$, and $5\pi/6$ between the roots, or equivalently, the angles $\pi/3$, $\pi/4$, and $\pi/6$ between the reflection mirrors. The absence of a direct link between two nodes implies that

A_n $(n \geq 1)$	B_n $(n \geq 3)$	C_n $(n \geq 2)$	D_n $(n \geq 4)$	E_6	E_7
$(n+1)!$	$2^n n!$	$2^n n!$	$2^{n-1} n!$	$2^7 3^4 5$	$2^{10} 3^4 5 7$
E_8	F_4	G_2	$H_2(m)$	H_3	H_4
$2^{14} 3^5 5^2 7$	$2^7 3^2$	12	$2m$	120	120^2

TAB. 2.1. Orders of the finite Coxeter groups.

the corresponding simple roots, as well as the mirrors, are orthogonal. Note that the relative angles of the mirrors of B_n and C_n coincide. Hence their W-groups are isomorphic. Their simple roots differ by length.

We adopt the Dynkin numbering of nodes. The numbering proceeds from left to right $1, 2, \ldots$ In case of D_n and E_6, E_7, E_8, the node above the main line has the highest number, respectively n, 6, 7, 8.

Orders of the finite Coxeter groups are provided in Table 1 for groups with connected diagrams. When a diagram has several disconnected components, the order is the product of orders corresponding to each subdiagram.

2.3. Root and weight lattices

Information essentially equivalent to that provided by the Coxeter and Dynkin diagrams is also given in terms of $n \times n$ matrices C, called the Cartan matrices. Relative angles and lengths of simple roots can be used to form the Cartan matrix for each group. Its matrix elements are calculated as

$$C = (C_{jk}) = \left(\frac{2\langle \alpha_j, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} \right), \qquad j, k \in \{1, 2, \ldots, n\}\,. \tag{2.3}$$

Cartan matrices and their inverses are given in many places, e.g., [**21, 8**].

The Cartan matrices can be defined for any finite Coxeter group by using formula (2.3). For non-crystallographic groups the matrices are

$$C(H_2) = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}, \quad C(H_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}, \quad C(H_4) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix},$$

where τ is the larger of the solutions of the algebraic equation $x^2 = x + 1$, i.e. $\tau = \frac{1}{2}(1 + \sqrt{5})$.

In addition to the basis of simple roots (α-basis), it is useful to introduce the basis of fundamental weights (ω-basis). Subsequently, most of our computations will be performed in the ω-basis.

$$\alpha = C\omega\,, \qquad \omega = C^{-1}\alpha\,.$$

Note the important relation:

$$\langle \alpha_k, \omega_j \rangle = \delta_{jk}\frac{\langle \alpha_k, \alpha_k \rangle}{2}\,, \qquad j,k \in \{1,2,\dots,n\}\,. \tag{2.4}$$

Illustrations showing the α- and ω-bases of A_2, C_2, and G_2 are given in Figure 1 of [**45**].

The root lattice Q and the weight lattice P of G are formed by all integer linear combinations of simple roots, respectively fundamental weights, of G,

$$Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n\,, \qquad P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n\,. \tag{2.5}$$

Here $\mathbb{Z}$ stands for any integer. For the groups that have simple roots of two different lengths, one may define the root lattice of n linearly independent short roots, which cannot all be simple. In general, $Q \subseteq P$, with $Q = P$ only for E_8, F_4, and G_2.

If G is one of the non-crystallographic Coxeter groups, the lattices Q and P are dense everywhere.

Since α- and ω-bases are not orthogonal and not normalized, it is sometimes useful to work with orthonormal bases. For crystallographic groups, they are found in many places, for example [**7, 8**]. For non-crystallographic groups, H_2, H_3 and H_4; see [**10, 44**].

2.4. The orbits of Coxeter groups

2.4.1. Computing points of an orbit

Given the reflections r_α, $\alpha \in \Pi(\alpha)$, of a Coxeter group G, and a seed point $\lambda \in \mathbb{R}^n$, the points of the orbit $G(\lambda)$ are given by the set of distinct points generated by repeated application of the reflections r_α to λ. All points of an orbit are equidistant from the origin. The radius of an orbit is the distance of (any) point of the orbit from the origin.

There are practically important considerations which make it almost imperative that the computation of the points of any orbit of G be carried out in the ω-basis, as follows:

- Every orbit contains precisely one point with nonnegative coordinates in the ω-basis. We specify the orbit by that point, calling it the dominant point of the orbit.
- Given a dominant point λ of the group G in the ω-basis, one readily finds the size of the orbit $G(\lambda)$, i.e. the number of points in the orbit, using the order $|G|$ of the Coxeter group and the order of the stabilizer of λ in G:

$$|G(\lambda)| = \frac{|G|}{|\operatorname{Stab}_G(\lambda)|} \tag{2.6}$$

  Here $\operatorname{Stab}_G(\lambda)$ is a Coxeter subgroup of G. To find it, one needs to attach the ω-coordinates of λ to the corresponding nodes of the diagram of G. The subdiagram carrying the coordinates 0 is the diagram of $\operatorname{Stab}_G(\lambda)$.
- Due to (2.4), the reflections (2.1) are particularly simple when applied to ω's:

$$r_k\omega_j = \omega_j - \frac{2\langle\alpha_k, \omega_j\rangle}{\langle\alpha_k, \alpha_k\rangle}\alpha_k = \omega_j - \delta_{jk}\alpha_k\,. \tag{2.7}$$

- Starting from the dominant point of an orbit, it suffices to apply, during the computation of the orbit points, only reflections corresponding to positive coordinates of any given weight. All points of the orbit are found in this way.

2.4.2. Orbits of A_2, C_2, G_2, and H_2

We give some examples of orbits. Let $a, b > 0$ be the coordinates in $\omega-$basis.

$$\begin{aligned}
A_2\ :\quad G((a,0)) =& \{(a,0),\ (-a,a),\ (0,-a)\}\,,\\
G((0,b)) =& \{(0,b),\ (b,-b),\ (-b,0)\}\,,\\
G((a,b)) =& \{(a,b),\ (-a,a+b),\ (b,-a-b),\ (a+b,-b)\,,\\
& (-a-b,a),\ (-b,-a)\}\,.
\end{aligned}$$

In particular, the orbit $G((1,1)) = \{\pm(1,1), \pm(-1,2), \pm(2,-1)\}$ consists of the vertices of a regular hexagon of radius $\sqrt{2}$. It is the root system of A_2.

$$\begin{aligned} C_2 \ : \quad G((a,0)) =& \{\pm(a,0),\ \pm(-a,a)\}\,, \\ G((0,b)) =& \{\pm(0,b),\ \pm(2b,-b)\}\,, \\ G((a,b)) =& \{\pm(a,b),\ \pm(-a,a+b),\ \pm(a+2b,-a-b), \\ & \pm(-a-2b,b)\}. \end{aligned}$$

In particular, the orbits $G((2,0))$ and $G((0,1))$ of radii $\sqrt{2}$ and 1 are respectively the vertices and midpoints of the sides of a square. Together the two orbits form the root system of C_2.

$$\begin{aligned} G_2 \ : \quad G((a,0)) =& \{\pm(a,0),\ \pm(-a,3a),\ \pm(2a,-3a)\}\,, \\ G((0,b)) =& \{\pm(0,b),\ \pm(b,-b),\ \pm(-b,2b)\}\,, \\ G((a,b)) =& \{\pm(a,b),\ \pm(-a,3a+b),\ \pm(2a+b,-3a-b), \\ & \pm(-2a-b,3a+2b),\ \pm(a+b,-3a-2b)\,, \\ & \pm(-a-b,b)\}\,. \end{aligned}$$

In particular, the orbits $G((1,0))$ and $G((0,1))$ are the vertices of regular hexagons of radii $\sqrt{2}$ and $2/\sqrt{3}$, rotated relatively by 30°, i.e. they form a hexagonal star. Together the 2 orbits form the root system of G_2. The points of $G((a, a\sqrt{3}/\sqrt{2}))$, $a > 0$, are the vertices of a regular dodecahedron of radius $\sqrt{2}a$.

$$\begin{aligned} H_2 \ : \quad G((a,0)) =& \{(a,0),\ (-a,a\tau),\ (a\tau,-a\tau),\ (-a\tau,a),\ (0,-a)\}\,, \\ G((0,b)) =& \{(0,b),\ (b\tau,-b),\ (-b\tau,b\tau),\ (b,-b\tau),\ (-b,0)\}\,, \\ G((a,b)) =& \{(a,b),\ (-a,b+a\tau),\ (a\tau+b\tau,-b-a\tau), \\ & (-a\tau-b\tau,a+b\tau),\ (b,-a-b\tau),\ (a+b\tau,-b), \\ & (-a-b\tau,a\tau+b\tau),\ (b+a\tau,-a\tau-b\tau), \\ & (-b-a\tau,a),\ (-b,-a)\}\,. \end{aligned}$$

		A_2	C_2	G_2	H_2	$H_2(7)$	1	2	3
1	●●	6	8	12	10	14	✓		
2	●○	3	4	6	5	7		✓	
3	○●	3	4	6	5	7			✓
4	⋆●	3	4	6	5	7	✓	✓	
5	●⋆	3	4	6	5	7	✓		✓

TAB. 2.2. Number of faces of $2D$ polytopes with Coxeter group symmetry. The first three rows specify representatives of G-orbits of $2D$ polytopes. A black (open) dot in the second column stands for a positive (zero) coordinate in the ω-basis of the dominant point representing the orbit of vertices. The number of vertices is listed in the subsequent five columns. Rows 4 and 5 refer to the edges of the polytopes. A star in the second column indicates the reflection generating the symmetry group of the edge. The number of edges is shown for each group in subsequent columns. Check marks in one of the last three columns indicate the faces which belong to the polytope described in that column.

In particular, the orbits $G((a,0))$ and $G((0,b))$ are the vertices of regular pentagons of radii $a\sqrt{2}$ and $b\sqrt{2}$, rotated relatively by 36°. The orbit $G((a,a))$ forms a regular decahedron. The orbit $G((\tau,\tau))$ consists of the roots of H_2.

An orbit of A_2 or H_2 contains, with every point (p,q) also the point $(-q,-p)$. Note that in the examples of this subsection the constants a and b do not need to be integers. All one requires is that they are positive. Effects of special choices of these constants are exemplified in (2.30) and (2.31) below.

2.4.3. Orbits of A_3, B_3, C_3, and H_3

We give some examples of orbits. Let $a, b, c > 0$ be the coordinates in $\omega-$basis.

$$
\begin{aligned}
A_3 \ : \quad G((a,0,0)) =& \{(a,0,0),\ (-a,a,0),\ (0,-a,a),\ (0,0,-a)\}\,,\\
G((0,b,0)) =& \{\pm(0,b,0),\ \pm(b,-b,b),\ \pm(-b,0,b)\}\,,\\
G((1,1,0)) =& \{(1,1,0),\ (-1,2,0),\ (2,-1,1),\ (1,-2,2),\ (-2,1,1),\\
& (2,0,-1),\ (-1,-1,2),\ (1,0,-2),\ (-2,2,-1),\\
& (-1,1,-2),\ (0,-2,1),\ (0,-1,-1)\}\,.
\end{aligned}
$$

$$
\begin{aligned}
B_3 \ : \quad G((a,0,0)) =& \{\pm(a,0,0),\ \pm(-a,a,0),\ \pm(0,-a,2a)\}\,,\\
G((0,b,0)) =& \{\pm(0,b,0),\ \pm(b,-b,2b),\ \pm(-b,0,2b),\ \pm(b,b,-2b),\\
& \pm(-b,2b,-2b),\ \pm(2b,-b,0)\}\,,\\
G((0,0,c)) =& \{\pm(0,0,c),\ \pm(0,c,-c),\ \pm(c,-c,c),\ \pm(c,0,-c)\}\,.
\end{aligned}
$$

$$
\begin{aligned}
C_3 \ : \quad G((a,0,0)) =& \{\pm(a,0,0),\ \pm(-a,a,0),\ \pm(0,-a,a)\}\,,\\
G((0,b,0)) =& \{\pm(0,b,0),\ \pm(b,-b,b),\ \pm(-b,0,b),\ \pm(b,b,-b),\\
& \pm(-b,2b,-b),\ \pm(2b,-b,0)\}\,,\\
G((0,0,c)) =& \{\pm(0,0,c),\ \pm(0,2c,-c),\ \pm(2c,-2c,c),\ \pm(2c,0,-c)\}\,.
\end{aligned}
$$

$$
\begin{aligned}
H_3 \ : \quad G((a,0,0)) =& \{\pm(a,0,0),\ \pm(-a,a,0),\ \pm(0,-a,a\tau),\ \pm(0,a\tau,-a\tau),\\
& \pm(a\tau,-a\tau,a),\ \pm(-a\tau,0,a)\}\,,\\
G((0,0,c)) =& \{\pm(0,0,c),\ \pm(0,\tau c,-c),\pm(\tau c,-\tau c,\tau^2 c),\pm(-\tau c,0,\tau^2 c),\\
& \pm(a\tau,-a\tau,a),\ \pm(-a\tau,0,a)\}\,.
\end{aligned}
$$

	Diagram	A_3	B_3	C_3	H_3	1	2	3	4	5	6	7
1	$\bullet\bullet\bullet$	24	48	48	120	$\checkmark$						
2	$\bullet\bullet\circ$	12	24	24	60		$\checkmark$					
3	$\bullet\circ\bullet$	12	24	24	60			$\checkmark$				
4	$\circ\bullet\bullet$	12	24	24	60				$\checkmark$			
5	$\bullet\circ\circ$	4	6	6	12					$\checkmark$		
6	$\circ\bullet\circ$	6	12	12	30						$\checkmark$	
7	$\circ\circ\bullet$	4	8	8	20							$\checkmark$
8	$\star\bullet\bullet$	12	24	24	60	$\checkmark$		$\checkmark$				
9	$\bullet\star\bullet$	12	24	24	60	$\checkmark$	$\checkmark$		$\checkmark$		$\checkmark$	
10	$\bullet\bullet\star$	12	24	24	60	$\checkmark$		$\checkmark$				
11	$\star\bullet\circ$	6	12	12	30		$\checkmark$			$\checkmark$		
12	$\circ\bullet\star$	6	12	12	30				$\checkmark$			$\checkmark$
13	$\star\star\bullet$	4	8	8	20	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	
14	$\star\bullet\star$	6	12	12	30	$\checkmark$		$\checkmark$				
15	$\bullet\star\star$	4	6	6	12	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$		$\checkmark$	$\checkmark$

TAB. 2.3. The $3D$ polytopes generated by a Coxeter group from a single seed point, and the number of their faces of dimension 0, 1 and 2. Decorated diagrams, rows 1 to 7, specify the polytopes. The dimension of a face equals the number of stars in the diagram. See 5.1 for additional explanations.

2.5. Orbits as polytopes

In this section, we recall an efficient method **[9]** of description for reflection-generated polytopes in any dimension.

The idea of the method consists in the following. Suppose we have an orbit $G(\lambda)$. Consider its points as vertices (faces of dimension 0) of the polytope also denoted $G(\lambda)$ in $\mathbb{R}^n$. Then for any face f of dimension $0 \le d \le n-1$, we identify its stabilizer $\mathrm{Stab}_{G(\lambda)}(f)$ in G, which is a product of two Coxeter subgroups of G:

$$\mathrm{Stab}_{G(\lambda)}(f) = G_1(f) \times G_2(D)$$

where $G_1(f)$ is the symmetry group of the face, and $G_2(D)$ stabilizes f pointwise, i.e. does not move it at all.

Our method consists in recursive decorations of the diagram of G, providing at each stage the subdiagrams of $G_1(f) = G(\star)$ and $G_2(D) = G(\circ)$ for faces of one type. The decoration of the nodes of the diagram indicates to which $G(\star)$ or $G(\circ)$ subgroups of the stabilizer the corresponding reflections belong. For further details, see [**9**]. A much wider application of this method is described in [**45, 46, 39**], including its exploitation in non Euclidean spaces.

We start with an extreme decoration of the diagram. It is equivalent to stating which coordinates of the dominant weight are positive relative to the ω-basis. The nodes are drawn as either open or black circles, i.e. zero or positive coordinates respectively.

Every possible extreme decoration fixes a polytope. There are only two rules for recursive decoration of the diagrams, starting from one of the extreme ones: (i) A single black circle is replaced by a star; (ii) open circles, that become adjacent to a star by diagram connectivity, are changed to black ones.

Tables 2 and 3 show the results of the application of the decoration rules for polytopes in $2D$ and $3D$ for all groups with connected diagrams. All polytopes for A_4, B_4, C_4, D_4, and H_4 are described in Tables 3 and 4 of [**9**].

2.5.1. Explanation of the Tables

A description of Table 2 is given in its caption.

Consider Table 3. The second column contains short-hand notation for several diagrams at once. We call them decorated diagrams. No links between nodes of a diagram are drawn because they would need to be different for each group in subsequent columns. The nodes do not reveal the relative lengths of roots, their decoration indicates to which of the pertinent subgroup of the stabilizer of G such a reflection belongs. Thus the diagrams of the second column of the Table apply to A_3, B_3, C_3 and H_3 at the same time.

Each line of the Table describes one of G-orbits of identical faces. The dimension of the face equals to the number of stars in its decorated diagram. Numerical entries in a row give the number of faces for polytopes of symmetry groups A_3, B_3, C_3 and H_3, shown in the header of the columns. The top seven rows show the starting decorations fixing the polytopes, and also the number of 0-faces (vertices) of the polytopes of each group. The check marks in one of the last seven columns indicate the faces belonging to the same polytope.

Example 2.5.1. As an example of how to read properties of polytope faces, consider rows number 5 and 2. The diagram in row 5 conveys the fact that $\lambda = a\omega_1$ with $a > 0$. The exact value of a affects only the size of the polytope, not its shape. The stabilizer of λ in the row 5 is given by the subdiagram of open circles, i.e. r_2 and r_3 generate its stabilizer. For A_3 the subdiagram is of type A_2, while for B_3 and C_3 it is of type C_2, and for H_3 it is of type H_2. Hence in row 5 the entries give the number of vertices as 24/6, 48/8, 48/8, 120/10 respectively.

The check mark in column 5 and row 5 indicates that faces belonging to our polytope are indicated by other check marks in column 5, namely in rows 11 and 13. The diagram of row 11 has just one star, hence the face is 1-dimensional (an edge). Its stabilizer (the subdiagram of stars and open circles) is of type $A_1 \times A_1$ for all four cases. Hence the number of edges is 24/4 for A_3, 48/4 for B_3 and C_3, and 120/4 for H_3. The only type of $2D$ face is given in row 13. The symmetry group of the face is generated by r_1 and r_2. It is of type A_2 for all four cases. Thus there are 24/6 faces in A_3, 48/6 in B_3 and C_3, and 120/6 in H_3 polytope.

Similarly, row 2 indicates that $\lambda = a\omega_1 + b\omega_2$, $a, b > 0$. It is stabilized by the group generated by r_3, which is of type A_1 for all four cases. Hence the number of vertices equals half of the order of the corresponding Coxeter group. There are two orbits of edges given in rows 9 and 11, while the two orbits of $2D$ faces are given by the check marks in rows 13 and 15.

Example 2.5.2. The $2D$ faces can actually be constructed knowing their symmetry and the seed point, say $(a, 0, 0)$. The diagram of the $2D$ face is $\star\ \star\ \bullet$, meaning that the symmetry group of the face is generated by r_1 and r_2. Moreover, it is of the same type (A_2) for all four groups. Then there are just three distinct vertices of the $2D$ face:

$$(a,0,0), \quad r_1(a,0,0), \quad r_2r_1(a,0,0)\,.$$

The $2D$ face is formed from the seed point $(a,0,0)$ by application of reflections r_1 and r_2.

The vertices of the $2D$ face are different triangles for each group, because they are given in their respective ω-basis:

$$\begin{aligned}
A_3 &: \quad (a,0,0),\ (-a,a,0),\ (0,-a,a),\\
B_3 &: \quad (a,0,0),\ (-a,a,0),\ (0,-a,2a),\\
C_3 &: \quad (a,0,0),\ (-a,a,0),\ (0,-a,a),\\
H_3 &: \quad (a,0,0),\ (-a,a,0),\ (0,-a,a\tau)\,.
\end{aligned}$$

Example 2.5.3. Let us consider row 2 in further detail. The starting point is $\lambda = a\omega_1 + b\omega_2$, where $a,b > 0$. There are two orbits of edges given by their endpoints:

$$((a,b,0), r_1(a,b,0))\,, \qquad ((a,b,0), r_2(a,b,0))\,,$$

and two orbits of $2D$ faces. Consider just the H_3 case. The $2D$ face of row 13 has the symmetry group generated by r_1, r_2 (A_2 type). It is a hexagon:

$$\begin{aligned}
&(a,b,0), \quad (-a,a+b,0), \quad (a+b,-b,\tau b), \quad (b,-a-b,\tau(a+b)),\\
&(-a-b,a,\tau b), \quad (-b,-a,\tau(a+b)).
\end{aligned}$$

The $2D$ face of row 15 has its symmetry group generated by r_2, r_3 (H_2 type). It is a pentagon:

$$\begin{aligned}
&(a,b,0), \quad (a+b,-b,\tau b), \quad (a+b,\tau b,-\tau b),\\
&(a+\tau^2 b,-\tau b,b), \quad (a+\tau^2 b,0,-b)\,.
\end{aligned}$$

In particular, when $a = b$, the pentagon and the hexagon are both regular. The polytope is then the familiar fullerene or ‘soccer ball’.

Further questions about the structure of polytopes can be answered within our formalism: How many $2D$ faces meet in a vertex? Which $2D$ faces meet in an edge? The higher the dimension, the more questions like these can be asked and answered. For more information on such questions and others (e.g. dual pairs of polytopes), we refer to [**9**].

2.6. Decomposition of products of polytopes

2.6.1. Multiplication of G-invariant polynomials

The product of G-polytopes together with its decomposition, as defined in 6.2 below, can be simply motivated by its correspondence to the product of more familiar objects than orbits, namely G-invariant polynomials, say $P(\lambda;x)$ and $P(\mu;x)$. Here λ and μ are dominant points of their orbits and x stands for n auxiliary independent variables $x_1, x_2, \dots, x_n$ whose nature is of no concern to us here. They can be thought of as, for example, complex or real variables. We introduce them in order to make sense of the definitions below.

Denote by $\lambda^{(i)} \in G(\lambda)$ the points of the orbit $G(\lambda)$, and by $\mu^{(k)} \in G(\mu)$, where

$$\lambda^{(i)} = \sum_{p=1}^{n} a_p^{(i)} \omega_p\,, \quad \mu^{(k)} = \sum_{q=1}^{n} b_q^{(k)} \omega_q\,, \qquad 1 \le i \le |G(\lambda)|\,, \quad 1 \le k \le |G(\mu)|\,. \tag{2.8}$$

Here $|G(\lambda)|$ and $|G(\mu)|$ denote the number of points in their orbits. Then we can introduce the polynomials:

$$P(\lambda;x) = \sum_{\lambda^{(i)} \in G(\lambda)} x^{\lambda^{(i)}} := \sum_{i=1}^{|G(\lambda)|} x_1^{a_1^{(i)}} x_2^{a_2^{(i)}} \cdots x_n^{a_n^{(i)}}\,,$$

$$P(\mu;x) = \sum_{\mu^{(k)} \in G(\mu)} x^{\mu^{(k)}} := \sum_{k=1}^{|G(\mu)|} x_1^{b_1^{(k)}} x_2^{b_2^{(k)}} \cdots x_n^{b_n^{(k)}}\,,$$

and their product,

$$P(\lambda;x) \otimes P(\mu;x) = \sum_{i=1}^{|G(\lambda)|} \sum_{k=1}^{|G(\mu)|} x_1^{a_1^{(i)}+b_1^{(k)}} x_2^{a_2^{(i)}+b_2^{(k)}} \cdots x_n^{a_n^{(i)}+b_n^{(k)}}\,. \tag{2.9}$$

The latter consists of the sum of $|G(\lambda)||G(\mu)|$ monomials which can be decomposed into the sum of polynomials defined by one G-orbit each.

Finally, consider an example: Let G be the group A_2, and $\lambda = (1,0)$ and $\mu = (0,1)$. Therefore $P((1,0);x) = x_1 + x_1^{-1}x_2 + x_2^{-1}$ and $P((0,1);x) = x_2 + x_1x_2^{-1} + x_1^{-1}$. Their products decompose as follows,

$$
\begin{aligned}
P((1,0);x)\otimes P((0,1);x) = & \{x_1x_2 + x_1^2x_2^{-1} + x_1^{-1}x_2^2 + x_1^{-2}x_2 + x_1x_2^{-2} + x_1^{-1}x_2^{-1}\} \\
& +3 = P((1,1);x) + 3P((0,0);x)\,, \\
P((1,0);x)\otimes P((1,0);x) = & \{x_1^2 + x_1^{-2}x_2^2 + x_2^{-2}\} + 2\{x_2 + x_1x_2^{-1} + x_1^{-1}\} \\
= & P((2,0);x) + 2P((0,1);x)\,.
\end{aligned}
$$

2.6.2. Products of G-orbits

Suppose we are given two orbits, say $G(\lambda)$ and $G(\mu)$, of the same Coxeter group G. Let $\lambda^{(i)}$ and $\mu^{(k)}$ be the points of $G(\lambda)$ and $G(\mu)$ respectively, numbered in some way. We define the product of two orbits as

$$
G(\lambda)\otimes G(\mu) := \bigcup_{\lambda^{(i)}\in G(\lambda),\ \mu^{(k)}\in G(\mu)} (\lambda^{(i)} + \mu^{(k)})\,. \tag{2.10}
$$

The left side is obviously G-invariant, therefore the right side is also G-invariant. Hence it can be decomposed into a union of several G-orbits. The highest and the lowest components of such a decomposition are easily obtained:

$$
G(\lambda)\otimes G(\mu) = G(\lambda+\mu)\cup\cdots\cup G(\lambda+\overline{\mu}). \tag{2.11}
$$

Here, $\lambda+\mu$ is the sum of the dominant points of the orbits $G(\lambda)$ and $G(\mu)$. The symbol $\overline{\mu}$ stands for the unique lowest point of $G(\mu)$ (all coordinates are non-positive in the ω-basis). Frequently, it happens that $\lambda+\overline{\mu}$ is not a dominant point, i.e. the highest point in its orbit, but it still identifies the orbit uniquely. Note also that $\lambda+\overline{\mu}$ and $\mu+\overline{\lambda}$ always belong to the same G-orbit. The lowest component $G(\lambda+\overline{\mu})$ often appears more than once in the decomposition.

For a geometric interpretation of (2.11), recall that all orbits in (2.11) are concentric, having the origin as their common center, and that points of one orbit are equidistant from the origin. In physics, the product on the left side of (2.11) can be thought of as a certain 'interaction' between two orbit-layers, resulting on the right side in an 'onion'-like structure of several concentric orbit-layers.

To simplify the notation in the following examples, we write just λ instead of $G(\lambda)$, so that $\lambda \otimes \mu$ means $G(\lambda) \otimes G(\mu)$.

2.6.3. Two-dimensional examples

$$
\begin{aligned}
A_2 \ : \quad & (1,0) \otimes (0,1) = (1,1) \cup 3(0,0)\,, \\
& (1,0) \otimes (1,1) = (2,1) \cup 2(1,0) \cup 2(0,2)\,, \\
& (1,1) \otimes (1,1) = (2,2) \cup 2(1,1) \cup 2(3,0) \cup 2(0,3) \cup 6(0,0)\,. \\
C_2 \ : \quad & (1,0) \otimes (0,1) = (1,1) \cup 2(1,0)\,, \\
& (1,0) \otimes (1,1) = (2,1) \cup 2(2,0) \cup 2(0,2) \cup 2(0,1)\,, \\
& (1,1) \otimes (1,1) = (2,2) \cup 2(2,1) \cup 2(4,0) \cup 2(2,0) \cup 2(0,3) \cup \\
& \qquad\qquad\qquad\quad 2(0,1) \cup 8(0,0)\,. \\
G_2 \ : \quad & (1,0) \otimes (0,1) = (1,1) \cup 2(0,2) \cup 2(0,1)\,, \\
& (1,0) \otimes (1,1) = (2,1) \cup (1,2) \cup (1,1) \cup 2(0,4) \cup 2(0,2) \cup 2(0,1)\,, \\
& (1,1) \otimes (1,1) = (2,2) \cup 2(1,1) \cup 2(1,3) \cup 2(3,0) \cup 2(2,0) \cup \\
& \qquad\qquad\qquad\quad 2(1,0) \cup 2(0,5) \cup 2(0,4) \cup 2(0,1) \cup 12(0,0)\,. \\
H_2 \ : \quad & (1,0) \otimes (0,1) = (1,1) \cup (\tau-1,\tau-1) \cup 5(0,0)\,, \\
& (1,0) \otimes (1,1) = (2,1) \cup (\tau,\tau-1) \cup (\tau-1,1) \cup 2(1,0) \cup \\
& \qquad\qquad\qquad\quad 2(0,\tau+1)\,, \\
& (1,1) \otimes (1,1) = (2,2) \cup 2(\tau,\tau) \cup 2(\tau-1,\tau-1) \cup 2(2+\tau,0) \cup \\
& \qquad\qquad\qquad\quad 2(2\tau-1,0) \cup 2(0,2+\tau) \cup 2(0,2\tau-1) \cup 10(0,0)\,.
\end{aligned}
$$

2.6.4. Three-dimensional examples

$$
\begin{aligned}
A_3 \ : \quad & (1,0,0)\otimes(0,0,1) = (1,0,1)\cup 4(0,0,0)\,,\\
& (1,0,1)\otimes(0,1,0) = (1,1,1)\cup 3(2,0,0)\cup 4(0,1,0)\cup 3(0,0,2)\,,\\
& (1,1,0)\otimes(0,0,1) = (1,1,1)\cup 3(2,0,0)\cup 2(0,1,0)\,.\\
B_3 \ : \quad & (1,0,0)\otimes(0,0,1) = (1,0,1)\cup 3(0,0,1)\,,\\
& (1,0,1)\otimes(0,1,0) = (1,1,1)\cup 2(2,0,1)\cup 3(1,0,1)\cup 2(0,1,1)\cup\\
& \qquad\qquad\qquad\qquad 3(0,0,3)\cup 6(0,0,1)\,,\\
& (1,1,0)\otimes(0,0,1) = (1,1,1)\cup 2(2,0,1)\cup 2(1,0,1)\cup 2(0,1,1)\,.\\
C_3 \ : \quad & (1,0,0)\otimes(0,0,1) = (1,0,1)\cup 2(0,1,0)\,,\\
& (1,0,1)\otimes(0,1,0) = (1,1,1)\cup 2(2,1,0)\cup 2(1,0,1)\cup 4(2,0,0)\cup\\
& \qquad\qquad\qquad\qquad 4(0,2,0)\cup 4(0,1,0)\cup 3(0,0,2)\,,\\
& (1,1,0)\otimes(0,0,1) = (1,1,1)\cup 2(2,1,0)\cup 2(1,0,1)\cup 4(0,1,0)\,.\\
H_3 \ : \quad & (1,0,0)\otimes(0,0,1) = (1,0,1)\cup (0,\tau-1,\tau-1)\cup 5(\tau,0,0)\cup\\
& \qquad\qquad\qquad\qquad 3(0,0,\tau-1)\,.
\end{aligned}
$$

2.6.5. Decomposition of products of E_8 orbits.

We say that an orbit is fundamental if its dominant weight in the ω-basis has precisely one coordinate equal to 1 and all others are zero. Thus E_8 has 8 fundamental orbits. Their sizes range from 240 to over 17 000.

All 36 different products of fundamental orbits of E_8 were decomposed in **[13]** and are explicitly shown within the tables. They were indispensable in solving the main problem of **[13]**, namely the decomposition of products of fundamental representations of E_8.

2.7. Decomposition of symmetrized powers of orbits

2.7.1. Symmetrized powers of G-polynomials

The product of m identical polynomials, say $P(\lambda; x)$, is the subject of the action of the permutation group S_m of m elements. Thus it can be decomposed into a sum of components with a specific permutation symmetry. It is well known from representation theory that the permutation symmetry commutes with the action of the Weyl group. Consequently, each permutation symmetry component can be decomposed into a sum of polynomials.

Let $\square$ be short-hand notation for a polynomial (2.6.1). The product of two and more copies of $\square$ decomposes into the symmetry components indicated by their Young diagrams:

$$\square \otimes \square = \begin{array}{c}\square\\[-4pt]\square\end{array} + \square\square, \qquad \square \otimes \square \otimes \square = \begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array} + 2\begin{array}{l}\square\square\\[-4pt]\square\end{array} + \square\square\square, \quad \dots \tag{2.12}$$

In general, the square stands for a set of G-invariant items, each square containing the same items. Those can be monomials of a polynomial, or weights in the case of the weight system of a representation of a semisimple Lie group/algebra, or points of a G-orbit. The product of m copies of the same square decomposes into permutational symmetry components according to the representations of the group S_m. The components are identified by their Young diagram. Each of the components is further decomposable into the sum of parts that are labeled by the orbits of the Coxeter group G.

In order to perform such a two-step decomposition, (i) the items of the square need to be numbered consecutively in any convenient way. The items belonging to a particular permutation symmetry component are then determined according to the inequalities, shown in the next subsection, and more generally implied by the corresponding Young diagram. Then (ii) items belonging to a particular Young diagram, which are labeled by points transformed by G, are sorted out into the Coxeter group orbits. Practically it suffices to find the items labeled by dominant points.

2.7.2. Symmetrized powers of G-orbits

For simplicity of notation let us continue to label an orbit $G(\lambda)$ by its dominant point λ. The product of the same two G-orbits decomposes into its symmetric and antisymmetric parts:

$$\lambda \otimes \lambda = (\lambda^2)_{symm} \cup (\lambda^2)_{anti} \tag{2.13}$$

Each of the two terms of the right side is further decomposable into the sum of individual orbits. Let $\lambda_1, \lambda_2, \ldots$ be the points of the orbit λ numbered in any order. Then the content of the two parts is determined by the following inequalities

$$(\lambda^2)_{symm} \ni \lambda_p + \lambda_q\,, \quad p \geq q\,, \tag{2.14}$$

$$(\lambda^2)_{anti} \;\;\ni \lambda_p + \lambda_q\,, \quad p > q\,. \tag{2.15}$$

The product of 3 copies of λ decomposes likewise

$$\lambda \otimes \lambda \otimes \lambda = (\lambda^3)_{symm} \cup (\lambda^3)_{anti} \cup 2(\lambda^3)_{mixed}\,, \tag{2.16}$$

where permutation symmetry components are formed from the N points as follows:

$$(\lambda^3)_{symm} \ni \lambda_p + \lambda_q + \lambda_s\,, \quad p \geq q \geq s\,, \tag{2.17}$$

$$(\lambda^3)_{anti} \;\;\ni \lambda_p + \lambda_q + \lambda_s\,, \quad p > q > s\,, \tag{2.18}$$

$$(\lambda^3)_{mixed} \ni \lambda_p + \lambda_q + \lambda_s\,, \quad p \geq q \text{ and } p > s\,. \tag{2.19}$$

Similarly, any higher power decomposes into permutation symmetry components where each is a sum of individual orbits.

2.7.3. Two-dimensional examples

$$
\begin{aligned}
A_2 \ :\quad & (0,1)^2_{symm} = (1,0) \cup (0,2)\,,\\
& (0,1)^2_{anti} \ \ = (1,0)\,.\\
& (1,1)^2_{symm} = (2,2) \cup (1,1) \cup (3,0) \cup (0,3) \cup 3(0,0)\,,\\
& (1,1)^2_{anti} \ \ = (1,1) \cup (3,0) \cup (0,3) \cup 3(0,0)\,.\\
& (1,0)^3_{symm} = (1,1) \cup (3,0) \cup (0,0)\,,\\
& (1,0)^3_{anti} \ \ = (0,0)\,,\\
& (1,0)^3_{mixed} = (1,1) \cup 2(0,0)\,.\\
C_2 \ :\quad & (0,1)^2_{symm} = (2,0) \cup (0,2) \cup 2(0,0)\,,\\
& (0,1)^2_{anti} \ \ = (2,0) \cup 2(0,0)\,.\\
& (1,0)^3_{symm} = (1,1) \cup (3,0) \cup 2(1,0)\,,\\
& (1,0)^3_{anti} \ \ = (1,0)\,,\\
& (1,0)^3_{mixed} = (1,1) \cup 3(1,0)\,.\\
G_2 \ :\quad & (0,1)^2_{symm} = (1,0) \cup (0,2) \cup (0,1) \cup 3(0,0)\,,\\
& (0,1)^2_{anti} \ \ = (1,0) \cup (0,1) \cup 3(0,0)\,.\\
& (1,0)^3_{symm} = (1,3) \cup (3,0) \cup (2,0) \cup 3(1,0) \cup 2(0,3) \cup 2(0,0)\,,\\
& (1,0)^3_{anti} \ \ = (2,0) \cup 2(1,0) \cup 2(0,0)\,,\\
& (1,0)^3_{mixed} = (1,3) \cup 2(2,0) \cup 5(1,0) \cup 2(0,3) \cup 4(0,0)\,.\\
H_2 \ :\quad & (0,1)^2_{symm} = (\tau,0) \cup (0,2) \cup (0,\tau-1)\,,\\
& (0,1)^2_{anti} \ \ = (\tau,0) \cup (0,\tau-1)\,.\\
& (1,0)^3_{symm} = (2-\tau,1) \cup (1,\tau) \cup (3,0) \cup (\tau,0) \cup (0,\tau-1)\,,\\
& (1,0)^3_{anti} \ \ = (\tau,0) \cup (0,\tau-1)\,,\\
& (1,0)^3_{mixed} = (2-\tau,1) \cup (1,\tau) \cup 2(\tau,0) \cup 2(0,\tau-1)\,.
\end{aligned}
$$

2.7.4. Three-dimensional examples

$$
\begin{aligned}
A_3 \ :\quad & (1,0,0)^3_{symm} = (1,1,0) \cup (3,0,0) \cup (0,0,1)\,, \\
& (1,0,0)^3_{anti} \quad = (0,0,1)\,, \\
& (1,0,0)^3_{mixed} = (1,1,0) \cup 2(0,0,1)\,. \\
B_3 \ :\quad & (1,0,0)^3_{symm} = (1,1,0) \cup (3,0,0) \cup 3(1,0,0) \cup (0,0,2)\,, \\
& (1,0,0)^3_{anti} \quad = 2(1,0,0) \cup (0,0,2)\,, \\
& (1,0,0)^3_{mixed} = (1,1,0) \cup 5(1,0,0) \cup 2(0,0,2)\,. \\
C_3 \ :\quad & (1,0,0)^2_{symm} = (2,0,0) \cup (0,1,0) \cup 3(0,0,0)\,, \\
& (1,0,0)^2_{anti} \quad = (0,1,0) \cup 3(0,0,0)\,. \\
H_3 \ :\quad & (1,0,0)^2_{symm} = (2,0,0) \cup (0,1,0) \cup (0,\tau-1,0) \cup 6(0,0,0)\,, \\
& (1,0,0)^2_{anti} \quad = (0,1,0) \cup (0,\tau-1,0) \cup 6(0,0,0)\,.
\end{aligned}
$$

2.8. Congruence classes, indices, and anomaly numbers of polytopes

Here we introduce numerical characterizations of W-orbits, analogs of similar quantities known for irreducible representations of semisimple Lie groups, which proved particularly useful in their application.

2.8.1. Congruence classes

Inclusion among the lattices (2.5) is an important property of the Weyl group W. The weight lattice P can be understood as a union of several components, each isomorphic to the root lattice Q. The components are shifted relative to each other by some elements of P. An individual component consists of points belonging to one congruence class of P. The index of Q in P, denoted $|Z|$, is the number of distinct congruence classes in P. The value of $|Z|$ reflects other properties of G. For example, it is the order of the center of G, it is a common denominator of coordinates of all points of P when given in the basis of simple roots, etc. One has $|Z| > 1$ for all G but for the exceptional simple Lie groups of types E_8, F_4, and G_2.

The congruence number c is a number attached to points of P. The value of c is common to all points of the same congruence class. It can be defined in a number of equivalent ways. Our definition coincides with that of [**31**]. All points of any W-orbit belong to the same congruence class. Furthermore, orbits obtained from the decomposition of a product belong to the same congruence class, and their congruence number is the sum of the congruence numbers of the orbits of the multiplication. That is also true for the decomposition of symmetrized powers of orbits.

Let $x = (x_1, x_2, \ldots, x_n) \in P$ be a point to consider in the ω-basis. Its congruence number $c(x)$ is given by the following formulas :

$$
\begin{aligned}
A_n &\quad : \quad c(x) = \sum_{k=1}^{n} k x_k \quad \bmod (n+1) \\
B_n &\quad : \quad c(x) = x_n \quad \bmod 2 \\
C_n &\quad : \quad c(x) = \sum_{k=1}^{[\frac{n+1}{2}]} x_{2k-1} \quad \bmod 2 \\
D_n &\quad : \quad c(x) = (c_1(x) \quad \bmod 2\,,\ c_2(x) \quad \bmod 4), \\
&\qquad\ \ c_1(x) = x_{n-1} + x_n \\
&\qquad\ \ c_2(x) = \begin{cases} 2x_1 + 2x_3 + \cdots + 2x_{n-2} + (n-2)x_{n-1} + nx_n, & n \text{ odd} \\ 2x_1 + 2x_3 + \cdots + 2x_{n-3} + (n-2)x_{n-1} + nx_n, & n \text{ even} \end{cases} \\
E_6 &\quad : \quad c(x) = x_1 - x_2 + x_4 - x_5 \quad \bmod 3 \\
E_7 &\quad : \quad c(x) = x_4 + x_6 + x_7 \quad \bmod 2
\end{aligned}
\tag{2.20}
$$

For E_8, F_4 and G_2 there is only one congruence class, namely $c(x) = 0$ for all $x \in P$. Note also that the roots of any group belong to the congruence class $c(x) = 0$. Hence also the points of the root lattice of any group belong to the congruence class $c(x) = 0$.

The points of any single G-orbit belong to the same congruence class because the difference between any two points of the same orbit is an integer linear combination of simple roots, as can be derived from (2.7).

For the non-crystallographic groups, the congruence classes can be similarly defined, involving their appropriate irrationality. It is important to recall that, in these cases, P is a dense lattice. The coordinates of $x \in P$, relative to the ω-basis, are the numbers $a + \tau b$, with $a, b \in \mathbb{Z}$.

$$H_2 \quad : \quad c(x) = \tau x_1 + 2x_2 \mod 5, \quad \text{where } \tau = 3 \tag{2.21}$$

2.8.2. The second and higher indices

The second and higher indices were defined [**53**] for weight systems of irreducible finite dimensional representations of compact semisimple Lie groups. Extensive tables of indices of degree 0, 2 and 4 are found in [**35**]. The fact that a weight system is a union of several W-orbits suggests that the indices could be introduced for individual orbits. Moreover, we introduce them also for non-crystallographic Coxeter groups with the same formulas.

For any finite Coxeter group G, we define an index $I_\lambda^{(2k)}$ of degree $2k$ of a G-orbit $G(\lambda)$ by

$$I_\lambda^{(2k)} = \sum_{\mu \in G(\lambda)} \langle \mu, \mu \rangle^k = \langle \lambda, \lambda \rangle^k I_\lambda^{(0)}, \qquad k = 0, 1, 2, \dots, \tag{2.22}$$

because points of $G(\lambda)$ are equidistant from the origin. Clearly $I_\lambda^{(0)} = |G(\lambda)|$ is the number of points of the orbit $G(\lambda)$ given by (2.6).

Higher indices of products of two orbits, $G(\lambda_1) \otimes G(\lambda_2)$, are also useful in calculating the decompositions. Let r be the rank of G.

$$I^{(2k)}(G(\lambda_1) \otimes G(\lambda_2)) = I^{(2k)}_{\lambda_1 \otimes \lambda_2} = I^{(2k)}_{\lambda_1 + \lambda_2} + \cdots + I^{(2k)}_{\overline{\lambda_1 + \lambda_2}} \tag{2.23}$$

$$
\begin{aligned}
I^{(0)}_{\lambda_1\otimes\lambda_2} &= I^{(0)}_{\lambda_1}\, I^{(0)}_{\lambda_2} \\
I^{(2)}_{\lambda_1\otimes\lambda_2} &= I^{(2)}_{\lambda_1}\, I^{(0)}_{\lambda_2} + I^{(0)}_{\lambda_1}\, I^{(2)}_{\lambda_2} \\
&= I^{(0)}_{\lambda_1}\, I^{(0)}_{\lambda_2}\,(\langle\lambda_1,\lambda_1\rangle + \langle\lambda_2,\lambda_2\rangle) \\
I^{(4)}_{\lambda_1\otimes\lambda_2} &= I^{(4)}_{\lambda_1}\, I^{(0)}_{\lambda_2} + \frac{2(r+2)}{r} I^{(2)}_{\lambda_1}\, I^{(2)}_{\lambda_2} + I^{(0)}_{\lambda_1}\, I^{(4)}_{\lambda_2}
\end{aligned}
$$

Table 4 presents examples of indices of degree 0, 2, 4, 6 and 8 for individual orbits of A_2, C_2, G_2 and H_2.

2.8.3. Anomaly numbers

Triangle anomaly numbers were introduced in physics [**61, 15, 50**] as quantities assigned to irreducible representations of a few compact semisimple Lie groups and calculated from the weight systems of their representations. Constraints on possible models in particle physics were imposed in terms of admissible values of the anomaly numbers of representations involved in a particular model. Generalization of the concept to all compact semisimple Lie groups and to higher than third degree anomaly number originates in [**58**]. Our goal here is to show that the anomaly numbers can be used also for constituents of the weight systems of irreducible representations, namely for W-orbits and more generally, for the orbits $G(\lambda)$ of any finite Coxeter group.

The anomaly number $I^{(2k-1)}_{\lambda}$ of degree $2k-1$ of the orbit $G(\lambda)$ of the Coxeter group G is defined as follows,

$$
I^{(2k-1)}_{\lambda} = \sum_{\mu\in G(\lambda)} \langle\mu,u\rangle^{2k-1}\,, \qquad k=1,2,\dots\,, \tag{2.24}
$$

where u is a special vector passing through the origin. In particular, $I^{(1)}=0$ in all cases. The anomaly number of physics literature is $I^{(3)}$, therefore it is the only one we consider.

Frequently used property of $I^{(3)}$ is the decomposition of the product of two orbits, which is the analog of (2.8.2):

$$
I^{(3)}_{\lambda_1\otimes\lambda_2} = I^{(3)}_{\lambda_1}\, I^{(0)}_{\lambda_2} + I^{(0)}_{\lambda_1}\, I^{(3)}_{\lambda_2} = I^{(3)}_{\lambda_1+\lambda_2} + \cdots + I^{(3)}_{\lambda_1+\overline{\lambda_2}} \tag{2.25}
$$

A_2	$I^{(0)}$	$I^{(2)}$	$3I^{(4)}$	$9I^{(6)}$	$27I^{(8)}$
$(1,0)$	3	2	4	8	16
$(2,0)$	3	8	64	512	4096
$(1,1)$	6	12	72	432	2592
$(2,1)$	6	28	392	5488	76832

C_2	$I^{(0)}$	$I^{(2)}$	$2I^{(4)}$	$4I^{(6)}$	$8I^{(8)}$
$(1,0)$	4	2	2	2	2
$(0,1)$	4	4	8	16	32
$(2,0)$	4	8	32	128	512
$(0,2)$	4	16	128	1024	8192
$(1,1)$	8	20	100	500	2500
$(2,1)$	8	40	400	4000	40000

G_2	$I^{(0)}$	$I^{(2)}$	$3I^{(4)}$	$9I^{(6)}$	$27I^{(8)}$
$(0,1)$	6	4	8	16	32
$(1,0)$	6	12	72	432	2592
$(0,2)$	6	16	128	1024	8192
$(0,3)$	6	36	648	11664	209952
$(2,0)$	6	48	1152	27648	663552
$(1,1)$	12	56	784	10976	153664

H_2	$I^{(0)}$	$(3-\tau)I^{(2)}$	$(3-\tau)^2I^{(4)}$
$(1,0)$	5	10	20
$(2,0)$	5	40	320
$(1,1)$	10	$20(\tau+2)$	$40(\tau+2)^2$
$(2,1)$	10	$10(4\tau+10)$	$10(4\tau+10)^2$

TAB. 2.4. Examples of the indices $I^{(2k)}, k = 0, \ldots, 4$.

In general terms, the direction of u can be characterized as follows. Suppose W in (2.24) is the Weyl group of a compact simple Lie group G, and that G has a maximal reductive subgroup of type $U(1)\times G'$. Then the direction of u is given by the direction corresponding to $U(1)$ in the Euclidean space spanned by the roots of G.

The first question to answer is when such a maximal subgroup is present. For a complete list of the cases see below [**6**]:

$$
\begin{aligned}
&A_n \supset A_{n-1}\times U(1) && n\geq 2\\
&A_n \supset A_k\times A_{n-k-1}\times U(1) && n\geq 3\,,\quad 1\leq k\leq [\tfrac{n-1}{2}]\\
&B_n \supset B_{n-1}\times U(1) && n\geq 3\\
&C_n \supset A_{n-1}\times U(1) && n\geq 2\\
&D_n \supset A_{n-1}\times U(1) && n\geq 4\\
&D_n \supset D_{n-1}\times U(1) && n\geq 5\\
&E_6 \supset D_5\times U(1) &&\\
&E_7 \supset E_6\times U(1) &&
\end{aligned}
\tag{2.26}
$$

As long as each orbit of a given group contains with every weight also its negative, the anomaly numbers are equal to zero. Therefore the interesting cases that remain are found in A_n, D_{2k+1}, E_6, and E_7. In physics, however, only the anomaly numbers of $A_n\supset A_{n-1}\times U(1)$ are used so far.

Anomaly numbers of H_2, H_3, and H_4 are also defined by (2.24). In those cases, however, the direction of u has to be determined differently since there is no $U(1)$ subgroup. Instead, one can require that u be orthogonal to selected simple roots: α_1 for H_2, α_1 and α_2 for H_3, and α_1, α_2, and α_3 for H_4. Anomaly numbers for H_2 are zero for all orbits. They will be considered elsewhere [**30**], along with the anomaly numbers of other non-crystallographic groups.

2.9. Concluding remarks

(1) Useful and interesting objects may turn out to be G-orbits with each point decorated by a sign [**51**] according to the following rule. The dominant

point, say λ, and all points obtained from it by an even number of reflections generating G, carry a positive sign, while all points of the orbit obtained from λ by an odd number of reflections carry a negative sign. Let us call an S-orbit a decorated orbit of λ of G, while the orbits without the sign decoration, i.e. all positive signs, are called C-orbits of λ of G. In order to avoid ambiguities, it should be stipulated that λ of an S-orbit must have all coordinates positive in ω-basis.

Multiplication of such orbits follows simple rules:

$$C\text{-orbit} \times C\text{-orbit} \quad \longrightarrow \quad C\text{-orbits}, \tag{2.27}$$

$$C\text{-orbit} \times S\text{-orbit} \quad \longrightarrow \quad S\text{-orbits}, \tag{2.28}$$

$$S\text{-orbit} \times S\text{-orbit} \quad \longrightarrow \quad C\text{-orbits}. \tag{2.29}$$

In (2.27), all coefficients in the decomposition of the product are positive integers, while in (2.28) and (2.29), all such coefficients are integers, but not all may be positive.

The decomposition of many products of C-orbits with lowest nontrivial S-orbit can be directly inferred from the tables [**8**], using the Weyl character formula.

(2) In the examples, we often required that a G-orbit consist of points of the weight lattice P. Very few properties of the orbits would have been lost, had we instead allowed λ to be any point in $\mathbb{R}^n$. The congruence classes would not then be applicable.

Consider the following products of A_2 orbits as examples:

$$(a,0) \otimes (\varepsilon,0) \qquad = (a+\varepsilon,0) \cup (a-\varepsilon,\varepsilon)\,, \qquad 0<\varepsilon \ll 1\,,\ a \geq 1\,, \tag{2.30}$$

$$(a,0) \otimes (0,a+\varepsilon) = (a,a+\varepsilon) \cup (0,\varepsilon)\,, \qquad 0<\varepsilon \ll 1\,,\ a \gg 1\,. \tag{2.31}$$

The radii of the two orbits in the decomposition (2.30) can be drawn arbitrarily close (in the sense of the continuous limit) by a suitable choice of ε, and in (2.31) they can be pushed as far apart as desired by the choice of a. The second orbit in (2.31) has a radius equal to $\varepsilon\sqrt{\frac{2}{3}}$.

(3) For a geometric interpretation of orbits as polytopes, refer to the paragraph following equation (2.11). The 'interaction' (i.e. product) between two concentric orbit-layers results in the layered structure of orbits. They are subject to the equality of indices of various degrees, congruence numbers, relations between anomaly numbers. Speculative interpretation can go further: Consider $I^{(2)}_{\lambda}$ as the 'energy' of the orbit and $I^{(2)}_{\lambda\otimes\lambda'}$ as the 'energy' of the interacting pair, etc.

(4) Although we did not pursue it here, orbit multiplication can be viewed as an 'interaction' between two orbits similarly as used in particle physics to view interacting multiplets of particles. A multiplet is described by the weight system of an irreducible representation of the corresponding Lie group/algebra. Here, the role of the multiplet would be given to the set of points of an orbit. In both cases, such interactions would be governed by the strict equality of indices of various degrees, congruence numbers, relations between anomaly numbers. But there is a price to pay for such a reinterpretation of multiplets: the overall invariance of the theory with respect to the Lie group would be reduced to the invariance with respect to the Coxeter group, or to its (discrete) image 'lifted' into the Lie group [**42**].

(5) It would be useful to ask additional questions about the properties of indices and anomaly numbers of various degrees. Such questions can be answered by adaptation of the methods used for the weight system of representations [**53, 58**].

(6) In place of finite Coxeter groups, we could have chosen to consider other finite groups for similar considerations [**34**]. The immediate motivations for our choice were recent applications in harmonic analysis, where W-orbits are playing a fundamental role. Equally interesting would be to consider orbits of infinite Coxeter groups. (A Coxeter group with connected diagram is of infinite order if its diagram is different from those listed in Section 2.) The orbits of representations of Kac-Moody algebras would be relatively easily amenable to such a study.

(7) Similarly, we could consider orbits of two or more seed points. A simple example is the root system of the group G_2. Choosing as the two seed points one short root and one long root, say α_2 and $\alpha_1 + 3\alpha_2$, the orbit of the pair is a star-like polygon formed by the root system of G_2.

(8) An interesting problem appears to be to pursue a similar study of orbits of the even subgroups of Coxeter groups, particularly because these subgroups are not Coxeter groups in general.

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Chapter 3

SIX TYPES OF $E-$FUNCTIONS OF THE LIE GROUPS $O(5)$ AND $G(2)$

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Abstract: New families of E-functions are described in the context of the compact simple Lie groups $O(5)$ and $G(2)$. These functions of two real variables generalize the common exponential functions and for each group, only one family is currently found in the literature. All the families are fully characterized, their most important properties are described, namely their continuous and discrete orthogonalities and decompositions of their products.

3.1. Introduction

We consider six infinite families of special functions which can be derived from the compact simple Lie groups of types

$$B_n\,,\quad C_n\,,\quad G_2\,,\quad F_4\,,\qquad 2\leq n<\infty\,. \tag{3.1}$$

These are all the simple Lie groups with roots of two different lengths. One of the six families has been described previously [**25**]. The other families are new.

Within each of the six family, the functions

(i) depend of n real variables, n being the rank of the underlying simple Lie group;

(ii) are periodic in various ways over the entire real Euclidean space $\mathbb{R}^n$;

(iii) are pairwise orthogonal when integrated over a finite ('fundamental') region of $\mathbb{R}^n$;

(iv) are pairwise orthogonal when their values, sampled at lattice fragment in the fundamental region, are summed up, the lattice being of any density and of symmetry dictated by the underlying group (3.1).

We limit our considerations to the Lie groups (3.1) of rank two, namely the groups $O(5) \equiv Sp(4)$ and $G(2)$. Their Lie algebras are denoted $B_2 \equiv C_2$ and G_2. There are neither principal nor technical obstacles to generalize the results of this paper to simple Lie groups of type (3.1) of ranks greater than 2.

A basic tool for defining the functions are the particular subgroups W^e, W^s, W^l of the finite reflection group W, which is the symmetry group of the root systems of the groups (3.1). These subgroups are called the even subgroups of W. They are of index 2 in W and it is seen from the orders of the reflection groups that they are not reflection generated groups.

In addition to the well known and extensively studied [**26, 24**] symmetric (C-) and antisymmetric (S-) functions of the Weyl group orbit, new families of W-orbit functions for the groups (3.1), called S^L- and S^S- , were recently discovered. In [**51**], functions defined using the subgroup W^e, called E-functions, were studied. Those functions are defined for every compact simple Lie group [**25, 23**] and they can be written as sums of symmetric and antisymmetric orbit functions (symbolically written as $E = C + S$). By considering the sums of pairs of the functions C, S, S^L, and S^S built using the same Weyl group we obtain 6 families of $E-$functions: We use the following notation for those functions

$$\begin{aligned} \Xi^{e+} &= C + S, & \Xi^{e-} &= S^L + S^S, \\ \Xi^{s+} &= C + S^S, & \Xi^{s-} &= S + S^S, \\ \Xi^{l+} &= C + S^L, & \Xi^{l-} &= S + S^L\,. \end{aligned} \tag{3.2}$$

In this paper we use a different approach to construct the $E-$functions using the sign homomorphisms (section 3.3.1).

General motivation [**57**] for the study of E-functions of two variables resides in the processing of digitally given data. The advantage of having a larger choice of families of functions is the fact that they are orthogonal in regions of different shapes, which can be more suitable for the particular data. Combined with the relative simplicity of these functions, one expects that the processing speed could be increased.

This paper is organized as follows. In section 2 we review some basic facts from the theory of Weyl groups. In section 3 the sign homomorphisms and their kernels are introduced. Sections 4 and 5 present in detail the even orbit function and the mixed even orbit function. The product of the $E-$functions is studied in the section 6.

3.2. Weyl groups and corresponding fundamental domains

3.2.1. Weyl group and affine Weyl group

The ordered set of simple roots $\Delta = (\alpha_1, \alpha_2)$ of a simple Lie algebra of rank 2 is a collection of 2 vectors spanning a real $2-$dimensional Euclidean space $\mathbb{R}^2$ [**21, 22**]. The simple roots of Δ form a basis of $\mathbb{R}^2$ satisfying certain specific conditions. These roots are specified by their lengths and the angle between them. Equivalently, the root system Δ can be determined either by the Coxeter–Dynkin diagram (and the corresponding Coxeter matrix M) or the Cartan matrix C of the simple Lie algebra. The coroots $\alpha_i^\vee$ are defined as $\alpha_i^\vee = 2\alpha_i / \langle \alpha_i, \alpha_i \rangle$, $i = 1, 2$. In addition to the $\alpha-$basis of simple roots, we define the weight $\omega-$basis by

$$\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}\,, \quad i, j \in \{1, 2\}.$$

The fourth basis, called the coweight $\omega^\vee-$basis, is given by $\omega_i^\vee = 2\omega_i / \langle \alpha_i, \alpha_i \rangle$, $i = 1, 2$.

We define the root lattice Q as the set of all integer linear combinations of simple roots

$$Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$$

and similarly we define the coroot $Q^\vee$ lattice by

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee.$$

Moreover, we define the weight lattice and the coweight lattice

$$P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \quad P^\vee = \mathbb{Z}\omega_1^\vee + \mathbb{Z}\omega_2^\vee.$$

Some important subsets of the weight lattice P are the cone of dominant weights P^+ and the cone of strictly dominant weights P^{++}:

$$P^+ = \mathbb{Z}^{\geq 0}\omega_1 + \mathbb{Z}^{\geq 0}\omega_2 \;\supset\; P^{++} = \mathbb{N}\omega_1 + \mathbb{N}\omega_2.$$

The reflection r_α, $\alpha \in \Delta$, which fixes the hyperplane orthogonal to α and passes through the origin can be explicitly written as $r_\alpha x = x - \langle \alpha, x\rangle \alpha^\vee$, where $x \in \mathbb{R}^2$.

Given a simple Lie algebra with the set of simple roots $\Delta = (\alpha_1, \alpha_2)$, the associated Weyl group W is a finite group generated by reflections $r_i \equiv r_{\alpha_i}$, $i = 1, 2$. The system of vectors $W\Delta$ ($W\Delta$ denotes W acting on the simple roots Δ) is the root system and contains the highest root $\xi \in W\Delta$. The affine reflection r_0 with respect to the highest root is given by

$$r_0 x = r_\xi x + \frac{2\xi}{\langle \xi, \xi\rangle}\,, \qquad r_\xi x = x - \frac{2\langle x, \xi\rangle}{\langle \xi, \xi\rangle}\xi\,, \qquad x \in \mathbb{R}^2\,.$$

By adding the affine reflection r_0 to the set of generators $\{r_1, r_2\}$ one obtains the affine Weyl group W^{aff}. The affine Weyl group W^{aff} consists of transformations of $\mathbb{R}^2$ from W and of shifts by vectors from the coroot lattice $Q^\vee$. In fact it holds that $W^{\mathrm{aff}} = Q^\vee \rtimes W$. The fundamental domain F of the action of W^{aff} on $\mathbb{R}^2$ is a triangle with vertices $\left\{0, \frac{\omega_1^\vee}{m_1}, \frac{\omega_2^\vee}{m_2}\right\}$, where m_1, m_2 are the coefficients of the highest root ξ in $\alpha-$basis, $\xi = m_1\alpha_1 + m_2\alpha_2$.

The set of dual roots $\Delta^\vee = (\alpha_1^\vee, \alpha_2^\vee)$ also generates a Weyl group W. The system of vectors $W\Delta^\vee$ is a root system and contains the highest dual root $\eta \in W\Delta^\vee$. The dual affine reflection $r_0^\vee$ with respect to the highest dual root is given by

$$r_0^\vee x = r_\eta x + \frac{2\eta}{\langle \eta, \eta\rangle}\,, \qquad r_\eta x = x - \frac{2\langle x, \eta\rangle}{\langle \eta, \eta\rangle}\eta\,, \qquad x \in \mathbb{R}^2\,.$$

By adding the dual affine reflection $r_0^\vee$ to the set of generators $\{r_1, r_2\}$ one obtains the dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$, see [**18**]. The dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ consists of transformations of $\mathbb{R}^2$ from W and of shifts by vectors from the root lattice Q; it holds that $\widehat{W}^{\mathrm{aff}} = Q \rtimes W$. The dual fundamental domain $F^\vee$ of the action of $\widehat{W}^{\mathrm{aff}}$ on $\mathbb{R}^2$ is a triangle with vertices $\left\{0, \frac{\omega_1}{m_1^\vee}, \frac{\omega_2}{m_2^\vee}\right\}$, where $m_1^\vee, m_2^\vee$ are the coefficients of the highest dual root η in $\alpha^\vee-$basis, $\eta = m_1^\vee\alpha_1^\vee + m_2^\vee\alpha_2^\vee$.

3.2.2. Fundamental domains

It is advantageous to distinguish explicitly the different root lengths in C_2 and G_2. Instead of the generic set of simple roots of rank 2 of the form $\Delta = (\alpha_1, \alpha_2)$, we use the following notation

$$\Delta(C_2) = (\alpha_s, \alpha_l)$$

$$\Delta(G_2) = (\alpha_l, \alpha_s).$$

The symbol α_s denotes the short simple root. For C_2 we use the standard normalization $\langle \alpha_s,\, \alpha_s \rangle = 1$ and for G_2 we have $\langle \alpha_s,\, \alpha_s \rangle = 2/3$. The symbol α_l denotes the long simple root. For C_2 and G_2 we have $\langle \alpha_l,\, \alpha_l \rangle = 2$. The angle between α_s and α_l is $3\pi/4$ for C_2 and $5\pi/6$ for G_2. The highest root ξ is equal to $2\alpha_s + \alpha_l$ for C_2 and to $2\alpha_l + 3\alpha_s$ for G_2. The ordering of the roots in bases $\Delta(C_2)$ and $\Delta(G_2)$ in (3.2.2), (3.2.2) is in accordance with the standard convention. Furthermore, we have the corresponding coroot $\alpha_s^\vee = 2\alpha_s / \langle \alpha_s, \alpha_s \rangle$, the weight ω_s satisfying $\langle \alpha_s^\vee,\, \omega_s \rangle = 1$, the coweight $\omega_s^\vee = 2\omega_s / \langle \alpha_s, \alpha_s \rangle$ and the corresponding reflection r_s (which we call short reflection). We define $\alpha_l^\vee$, ω_l, $\omega_l^\vee$ and r_l (long reflection) analogously. Therefore, the fundamental domains F have the following explicit form

$$F(C_2) = \{a\omega_s^\vee + b\omega_l^\vee \,|\, a,\, b \geq 0,\, 2a + b \leq 1\}$$

$$F(G_2) = \{a\omega_l^\vee + b\omega_s^\vee \,|\, a,\, b \geq 0,\, 2a + 3b \leq 1\}\,.$$

We also denote by X_s, X_l the lines ('mirrors') which are stabilized by r_s, r_l, i.e. orthogonal to α_s and α_l, respectively:

$$X_s = \left\{x \in \mathbb{R}^2 \,|\, \langle x,\, \alpha_s \rangle = 0\right\}, \quad X_l = \left\{x \in \mathbb{R}^2 \,|\, \langle x,\, \alpha_l \rangle = 0\right\}.$$

The lines which are stabilized with respect to the affine reflections r_0, $r_0^\vee$ are denoted by X_0, $X_0^\vee$, i.e.

$$X_0 = \left\{x \in \mathbb{R}^2 \,|\, \langle x,\, \xi \rangle = 1\right\}, \quad X_0^\vee = \left\{x \in \mathbb{R}^2 \,|\, \langle x,\, \eta \rangle = 1\right\}.$$

The segments of the lines X_s, X_l and X_0 which lie in the fundamental domain F are denoted by Y_s, Y_l and Y_0, respectively:

$$Y_s = X_s \cap F, \quad Y_l = X_l \cap F, \quad Y_0 = X_0 \cap F.$$

Analogously, we define

$$Y_s^\vee = X_s \cap F^\vee, \quad Y_l^\vee = X_l \cap F^\vee, \quad Y_0^\vee = X_0^\vee \cap F^\vee.$$

We distinguish the weights from the positive weight lattice P^+ which lie on the mirrors X_s, and X_l:

$$P^s = X_s \cap P^+, \quad P^l = X_l \cap P^+.$$

The fundamental domains F together with the root systems of C_2 and G_2 are depicted in Figure 3.1.

3.3. Homomorphisms and orbits

3.3.1. Sign homomorphisms

The Weyl group W can also be abstractly defined by the following presentation

$$r_i^2 = 1, \quad (r_i r_j)^{m_{ij}} = 1, \quad i,j = 1,2 \tag{3.3}$$

where integers m_{ij} denote elements of the Coxeter matrix.

Crucial for us are certain 'sign' homomorphisms $\sigma : W \to \{\pm 1\}$. An admissible σ has to satisfy the presentation condition (3.3)

$$\sigma(r_i)^2 = 1, \quad (\sigma(r_i)\sigma(r_j))^{m_{ij}} = 1, \quad i,j = 1,2. \tag{3.4}$$

There are two obvious choices $\mathfrak{1}$, σ^e of such sign homomorphisms which are defined for any $w \in W$

$$\mathfrak{1}(w) = 1$$

$$\sigma^e(w) = \det w.$$

It turns out that for root systems with two different lengths of roots there are two other choices available [**37**].

The Lie algebras C_2 and G_2 are the only simple Lie algebras of rank 2 which have two different lengths of roots. As such, they admit other sign homomorphisms (and corresponding special functions) besides the standard choices (3.3.1),

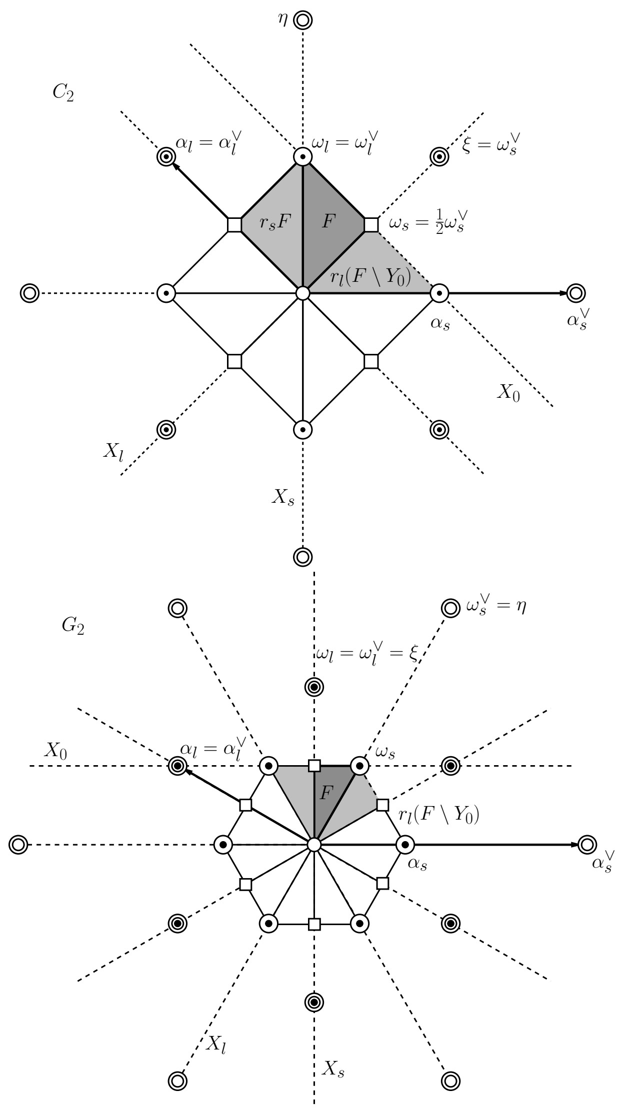

FIG. 3.1. The fundamental domains F and the root systems of C_2 and G_2; the circles with a small dot inscribed depict the roots of the root system $W\Delta$ and the circles with a smaller circle inside them depict the elements of the dual root system $W\Delta^\vee$.

(3.3.1). The Coxeter matrices M of C_2 and G_2 are the following:

$$M(C_2) = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}, \qquad M(G_2) = \begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}. \tag{3.5}$$

A sign homomorphism $\sigma : W \to \{\pm 1\}$ can be defined by prescribing its values on the generators r_s and r_l such that (3.4) is satisfied. The two obvious choices $\sigma(r_s) = \sigma(r_l) = 1$ and $\sigma^e(r_s) = \sigma^e(r_l) = -1$ lead to the standard homomorphisms

$\mathbb{1}$ and σ^e. For the rank 2 cases C_2 and G_2 the elements m_{12} of the Coxeter matrices (3.5) are even. Therefore, $\sigma(r_l)$ and $\sigma(r_s)$ can be independently ± 1 and (3.4) is still satisfied. Consequently, there are two more sign homomorphisms, which we denote by σ^s and σ^l:

$$\sigma^s(r_s) = -1\,, \quad \sigma^s(r_l) = 1\,, \tag{3.6}$$

$$\sigma^l(r_l) = -1\,, \quad \sigma^l(r_s) = 1\,. \tag{3.7}$$

Every element w from a Weyl group W can be written as a product of generators $w = r_{i_1} \dots r_{i_k}$ where $r_{i_j} \in \{r_s, r_l\}$. Equivalently, we reformulate the definition (3.6), (3.7):

$$\sigma^s(w) = \begin{cases} 1 & \text{if there is an even number of short reflections } r_s \text{ in } w \\ -1 & \text{if there is an odd number of short reflections } r_s \text{ in } w\,, \end{cases}$$

$$\sigma^l(w) = \begin{cases} 1 & \text{if there is an even number of long reflections } r_l \text{ in } w \\ -1 & \text{if there is an odd number of long reflections } r_l \text{ in } w\,. \end{cases}$$

3.3.2. Orbits and stabilizers

Except for the trivial homomorphism $\mathbb{1}$, the kernels $\ker \sigma$ of the sign homomorphisms σ form normal subgroups of index 2 in W. We denote the kernels of σ^e, σ^s and σ^l by W^e, W^s and W^l, respectively:

$$W^e = \{w \in W \,|\, \sigma^e(w) = 1\}\,,$$

$$W^s = \{w \in W \,|\, \sigma^s(w) = 1\}\,,$$

$$W^l = \left\{w \in W \,|\, \sigma^l(w) = 1\right\}.$$

Some general properties of the group W^e were derived in [**19**]. Explicit knowledge of the orbits and stabilizers of W^e, W^s and W^l will be needed for description of orbit functions. Generic orbits of the action of W^e, W^s and W^l on $\mathbb{R}^2$ for the cases of C_2 and G_2 are shown in Figures 3.2 and 3.3.

$$\begin{tikzcd}[column sep=small]
W^e & & (a,b) \arrow[dl, "r_l r_s"'] \arrow[dr, "r_s r_l"] & \\
& (a+2b,-a-b) \arrow[dr, "r_l r_s"'] & & (-a-2b,a+b) \arrow[dl, "r_s r_l"] \\
& & (-a,-b) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
W^s & (a,b) \arrow[dl, "r_s r_l r_s"'] \arrow[dr, "r_l"] & \\
(-a-2b,b) \arrow[dr, "r_l"'] & & (a+2b,-b) \arrow[dl, "r_s r_l r_s"] \\
& (-a,-b) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
W^l & (a,b) \arrow[dl, "r_s"'] \arrow[dr, "r_l r_s r_l"] & \\
(-a,a+b) \arrow[dr, "r_l r_s r_l"'] & & (a,-a-b) \arrow[dl, "r_s"] \\
& (-a,-b) &
\end{tikzcd}$$

FIG. 3.2. Orbits of the actions of the groups W^e, W^s and W^l of C_2. The coordinates (a,b) of the points in $\mathbb{R}^2$ are given in $\omega-$basis.

$$\begin{tikzcd}[column sep=small]
W^e & & (a,b) \arrow[dl, "r_s r_l"'] \arrow[dr, "r_l r_s"] & \\
& (2a+b,-3a-b) \arrow[d, "r_s r_l"'] & & (-a-b,3a+2b) \arrow[d, "r_l r_s"] \\
& (a+b,-3a-2b) \arrow[dr, "r_s r_l"'] & & (-2a-b,3a+b) \arrow[dl, "r_l r_s"] \\
& & (-a,-b) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
W^s & (a,b) \arrow[dl, "r_l"'] \arrow[dr, "r_s r_l r_s"] & \\
(-a,3a+b) \arrow[d, "r_s r_l r_s"'] & & (2a+b,-3a-2b) \arrow[d, "r_l"] \\
(a+b,-3a-2b) \arrow[d, "r_l"'] & & (-2a-b,3a+b) \\
(-a-b,b) & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
W^l & (a,b) \arrow[dl, "r_l r_s r_l"'] \arrow[dr, "r_s"] & \\
& & (a+b,-b) \arrow[d, "r_l r_s r_l"] \\
(-2a-b,3a+2b) \arrow[d, "r_s"'] & & \\
(a+b,-3a-2b) & & (-2a-b,3a+b) \arrow[d, "r_s"] \\
& & (a,-3a-b)
\end{tikzcd}$$

FIG. 3.3. Orbits of the actions of the groups W^e, W^s and W^l of G_2. The coordinates (a,b) of the points in $\mathbb{R}^2$ are given in $\omega-$basis.

For any $\lambda \in \mathbb{R}^2$ we have the stabilizer $\mathrm{Stab}_{\ker\sigma}(\lambda)$ of λ

$$\mathrm{Stab}_{\ker\sigma}(\lambda) = \{w \in \ker\sigma \mid w\lambda = \lambda\}$$

and we denote the orders of the stabilizers of W^e, W^s and W^l by d^e_λ, d^s_λ and d^l_λ, respectively

$$d^e_\lambda \equiv |\mathrm{Stab}_{W^e}(\lambda)|, \quad d^s_\lambda \equiv |\mathrm{Stab}_{W^s}(\lambda)|, \quad d^l_\lambda \equiv |\mathrm{Stab}_{W^l}(\lambda)|. \tag{3.8}$$

For our purposes, it is sufficient to consider only values of $\lambda \in P^+$. If $\lambda \in rP^+$, for $r = r_s, r_l$, then the stabilizers of λ and $r\lambda$ are conjugate and have the same order, i.e.

$$d^e_\lambda = d^e_{r\lambda}, \quad d^l_\lambda = d^l_{r\lambda}, \quad d^s_\lambda = d^s_{r\lambda}, \quad r = r_s, r_l.$$

The orders of stabilizers d^e_λ, d^s_λ and d^l_λ with $\lambda \in P^+$ are for the cases of C_2 and G_2 shown in Table 3.1.

$\lambda \in P^+$	C_2			G_2		
	d^e_λ	d^s_λ	d^l_λ	d^e_λ	d^s_λ	d^l_λ
(a,b)	1	1	1	1	1	1
$(a,0)$	1	2	1	1	1	2
$(0,b)$	1	1	2	1	2	1
$(0,0)$	4	4	4	6	6	6

TAB. 3.1. Orders of stabilizers of $\lambda \in P^+$ for C_2 and G_2, The coordinates (a,b) are in $\omega-$basis with $a \neq 0$, $b \neq 0$.

While calculating continuous orthogonality of various types of orbit functions, the number of elements in the Weyl group and the volume of the fundamental domain often appear. Let $|F|$ be the volume of the fundamental domain. We denote the number $|W||F|$ by K and we have (see e.g. **[18]**)

$$K \equiv |W||F| = \begin{cases} 2 & \text{for } C_2 \\ \sqrt{3} & \text{for } G_2. \end{cases} \tag{3.9}$$

3.3.3. Orbits and stabilizers on the maximal torus

The orbits and stabilizers of the maximal torus will be needed for the discrete calculus of orbit functions. We choose some arbitrary natural number M which will control the density of the grids appearing in this calculus **[18]**. The discrete calculus of orbit functions is performed over the finite group $\frac{1}{M}P^\vee/Q^\vee$. The finite complement set of weights is taken as the quotient group P/MQ. For $x \in \mathbb{R}^2/Q^\vee$, we denote the orbit of a subgroup $\ker\sigma$ by

$$(\ker\sigma)x = \left\{ wx \in \mathbb{R}^2/Q^\vee \mid w \in \ker\sigma \right\}.$$

The orders of the group stabilizers of W^e, W^s and W^l are denoted by ε^e, ε^s and ε^l:

$$\varepsilon^e(x) \equiv |W^e x|, \quad \varepsilon^s(x) \equiv |W^s x|, \quad \varepsilon^l(x) \equiv |W^l x|. \tag{3.10}$$

For practical purposes it is sufficient to determine the sizes of these orbits for the finite set

$$F_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F.$$

For a general review of these sets F_M see [**18**]. If $x \in rF_M$, for $r = r_s, r_l$, then the orbits of x and rx have identical size, i.e. for $x \in F_M$

$$\varepsilon^e(x) = \varepsilon^e(rx) \quad \varepsilon^s(x) = \varepsilon^s(rx), \quad \varepsilon^l(x) = \varepsilon^l(rx), \quad r = r_s, r_l.$$

For our cases C_2 and G_2 the sets F_M are of the following explicit form

$$\begin{aligned} F_M(C_2) &= \left\{ \frac{a}{M}\omega_s^\vee + \frac{b}{M}\omega_l^\vee \mid a,b,c \in \mathbb{Z}^{\geq 0}, c + 2a + b = M \right\} \\ F_M(G_2) &= \left\{ \frac{a}{M}\omega_l^\vee + \frac{b}{M}\omega_s^\vee \mid a,b,c \in \mathbb{Z}^{\geq 0}, c + 2a + 3b = M \right\}. \end{aligned}$$

The coefficients $\varepsilon^e(x)$, $\varepsilon^s(x)$ and $\varepsilon^l(x)$ for the cases C_2 and G_2 are listed in Table 3.2.

For any $\lambda \in P/MQ$, we denote the stabilizer $\mathrm{Stab}^\vee_{\ker\sigma}(\lambda)$ of λ by

$$\mathrm{Stab}^\vee_{\ker\sigma}(\lambda) = \{ w \in \ker\sigma \mid w\lambda = \lambda \}.$$

The corresponding orders of the stabilizers of the action of W^e, W^s and W^l on P/MQ are denoted by $h^{e\vee}_\lambda$, $h^{s\vee}_\lambda$ and $h^{l\vee}_\lambda$, respectively

$$h^{e\vee}_\lambda \equiv |\mathrm{Stab}^\vee_{W^e}(\lambda)|, \quad h^{s\vee}_\lambda \equiv |\mathrm{Stab}^\vee_{W^s}(\lambda)|, \quad h^{l\vee}_\lambda \equiv |\mathrm{Stab}^\vee_{W^l}(\lambda)|. \tag{3.11}$$

For practical purposes it is sufficient to determine the sizes of these stabilizers for the finite set

$$\Lambda_M \equiv P/MQ \cap MF^\vee.$$

$x \in F_M$	C_2			G_2		
	$\varepsilon^e(x)$	$\varepsilon^s(x)$	$\varepsilon^l(x)$	$\varepsilon^e(x)$	$\varepsilon^s(x)$	$\varepsilon^l(x)$
$[c, a, b]$	4	4	4	6	6	6
$[0, a, b]$	4	2	4	6	3	6
$[c, 0, b]$	4	4	2	6	3	6
$[c, a, 0]$	4	2	4	6	6	3
$[0, 0, b]$	1	1	1	2	1	2
$[0, a, 0]$	2	1	2	3	3	3
$[c, 0, 0]$	1	1	1	1	1	1

$\lambda \in \Lambda_M$	C_2			G_2		
	$h^{e\vee}_\lambda$	$h^{s\vee}_\lambda$	$h^{l\vee}_\lambda$	$h^{e\vee}_\lambda$	$h^{s\vee}_\lambda$	$h^{l\vee}_\lambda$
$[c, a, b]$	1	1	1	1	1	1
$[0, a, b]$	1	1	2	1	1	2
$[c, 0, b]$	1	1	2	1	2	1
$[c, a, 0]$	1	2	1	1	1	2
$[0, 0, b]$	2	2	4	2	2	2
$[0, a, 0]$	4	4	4	3	3	6
$[c, 0, 0]$	4	4	4	6	6	6

TAB. 3.2. Orders of orbits of $x \in F_M$ and stabilizers of $\lambda \in \Lambda_M$ for the cases C_2 and G_2. The coordinates $[c, a, b]$ of $x \in F_M(C_2)$, $x \in F_M(G_2)$ are as in (3.3.3), (3.3.3), respectively. The coordinates $[c, a, b]$ of $\lambda \in \Lambda_M(C_2)$ and $\lambda \in \Lambda_M(G_2)$ are taken from (3.3.3), (3.3.3), respectively. It is assumed that $a, b, c \neq 0$.

For a general review of the sets Λ_M see [**18**]. If $\lambda \in r\Lambda_M$, for $r = r_s, r_l$, then the stabilizers of λ and $r\lambda$ are conjugate and have identical size, i.e. for $\lambda \in \Lambda_M$ we have

$$h^{e\vee}_\lambda = h^{e\vee}_{r\lambda}, \quad h^{s\vee}_\lambda = h^{s\vee}_{r\lambda}, \quad h^{l\vee}_\lambda = h^{l\vee}_{r\lambda}, \quad r = r_s, r_l.$$

For C_2 and G_2, the sets Λ_M are of the following explicit form

$$\Lambda_M(C_2) = \left\{ a\omega_s + b\omega_l \mid a,b,c \in \mathbb{Z}^{\geq 0}, c+a+2b = M \right\}$$
$$\Lambda_M(G_2) = \left\{ a\omega_l + b\omega_s \mid a,b,c \in \mathbb{Z}^{\geq 0}, c+3a+2b = M \right\}.$$

The coefficients $h_\lambda^{e\vee}$, $h_\lambda^{s\vee}$ and $h_\lambda^{l\vee}$ for the cases C_2 and G_2 are listed in Table 3.2.

While calculating discrete orthogonality of various types of orbit functions, the number of elements of the Weyl group and the determinant of the Cartan matrix $\det C$ often appear. We denote the number $|W|\det C/2$ by k and we have (see e.g. [**18**])

$$k \equiv \frac{|W|\det C}{2} = \begin{cases} 8 & \text{for } C_2 \\ 6 & \text{for } G_2. \end{cases} \tag{3.12}$$

3.4. Even orbit functions

Any sign homomorphism $\sigma : W \to \{\pm 1\}$ determines, in general, a complex orbit function $\psi_\lambda^\sigma : \mathbb{R}^2 \to \mathbb{C}$ parametrized by $\lambda \in P$:

$$\psi_\lambda^\sigma(x) = \sum_{w\in W} \sigma(w)\, e^{2\pi i\langle w\lambda,\, x\rangle}, \quad x \in \mathbb{R}^2. \tag{3.13}$$

For the choice (3.3.1) of $\imath$ in (3.13), we obtain the so-called $C-$functions [**26**]. The $C-$functions $\psi_\lambda^{\imath}$ were studied in detail for all rank two cases in [**54, 55**]. For the choice (3.3.1) of σ^e, we obtain the well-known $S-$functions [**24**]. The $S-$functions $\psi_\lambda^{\sigma^e}$ resulting from homomorphism σ^e and (3.13) were described for all rank two cases in [**56**]. The remaining two options of homomorphisms (3.6), (3.7) and corresponding functions $\psi_\lambda^{\sigma^s}$, $\psi_\lambda^{\sigma^l}$, called S^l- and S^s-functions [**37**], were studied in detail for G_2 only recently in [**62**].

The kernels $\ker\sigma$ of the sign homomorphisms σ, $\sigma \neq \imath$, give rise to another type of 'even' orbit functions. They are complex functions $\Psi_\lambda^\sigma : \mathbb{R}^2 \to \mathbb{C}$ parametrized by $\lambda \in P$,

$$\Psi_\lambda^\sigma(x) = \sum_{w\in\ker\sigma} e^{2\pi i\langle w\lambda,\, x\rangle}, \quad x \in \mathbb{R}^2. \tag{3.14}$$

These 'even' orbit functions are invariant with respect to the action of $w \in \ker\sigma$

$$\Psi^\sigma_\lambda(wx) = \Psi^\sigma_\lambda(x)$$

$$\Psi^\sigma_{w\lambda}(x) = \Psi^\sigma_\lambda(x)$$

and the invariance with respect to shifts from $Q^\vee$ also holds

$$\Psi^\sigma_\lambda(x+q^\vee) = \Psi^\sigma_\lambda(x), \quad q^\vee \in Q^\vee. \tag{3.15}$$

In the following sections we investigate all the possible choices of sign homomorphisms and the resulting functions. We also discuss basic properties of those functions, including continuous and discrete orthogonality and corresponding transforms. It turns out that for all cases similar orthogonality relations to those in [**19, 43**] hold.

3.4.1. Ξ^{e+}−functions

For the choice σ^e, we obtain the so-called E−functions [**25**]. Here we denote these functions by the symbol $\Xi^{e+}_\lambda \equiv \Psi^{\sigma^e}_\lambda$ and the corresponding kernel is W^e. The invariance (3.4), (3.15) with respect to $Q^\vee \rtimes W^e$ allows us to consider Ξ^{e+}_λ only on the domain

$$F^{e+} = F \cup rF^\circ$$

where F° denotes the interior of F and r is an arbitrary generating reflection $r \in \{r_s, r_l\}$. Similarly, the invariance (3.4) restricts $\lambda \in P$ to the set

$$P_{e+} = P^+ \cup rP^{++}.$$

Thus, we have

$$\Xi^{e+}_\lambda(x) = \sum_{w\in W^e} e^{2\pi i\langle w\lambda,\, x\rangle}, \quad x \in F^{e+},\, \lambda \in P_{e+}.$$

These E−functions Ξ^{e+}_λ were studied for all rank two cases in detail in [**23**].

3.4.2. Ξ^{s+}−functions

We discuss the resulting functions $\Psi^{\sigma^s}_\lambda$ which correspond to the sign homomorphism σ^s. We denote these functions by $\Xi^{s+}_\lambda \equiv \Psi^{\sigma^s}_\lambda$ and the corresponding kernel is W^s. The invariance (3.4), (3.15) with respect to $Q^\vee \rtimes W^s$ allows us to consider Ξ_λ^{s+} only on the domain

$$F^{s+} = F \cup r_s(F \setminus Y_s).$$

Similarly, the invariance (3.4) restricts $\lambda \in P$ to the set

$$P_{s+} = P^+ \cup r_s(P^+ \setminus P^s).$$

Thus, we have

$$\Xi_\lambda^{s+}(x) = \sum_{w \in W^s} e^{2\pi i \langle w\lambda,\, x\rangle}, \quad x \in F^{s+},\ \lambda \in P_{s+}.$$

3.4.2.1. *Continuous orthogonality and $\Xi^{s+}-$transforms*

For any two weights $\lambda, \lambda' \in P_{s+}$ the corresponding $\Xi^{s+}-$functions are orthogonal on F^{s+}

$$\int_{F^{s+}} \Xi_\lambda^{s+}(x)\overline{\Xi_{\lambda'}^{s+}(x)}\, dx = K\, d_\lambda^s\, \delta_{\lambda\lambda'} \tag{3.16}$$

where d_λ^s, K are given by (3.8), (3.9), respectively. The $\Xi^{s+}-$functions determine symmetrized Fourier series expansions,

$$f(x) = \sum_{\lambda \in P_{s+}} c_\lambda^{s+} \Xi_\lambda^{s+}(x), \quad \text{where } c_\lambda^{s+} = \frac{1}{K d_\lambda^s} \int_{F^{s+}} f(x)\overline{\Xi_\lambda^{s+}(x)}\, dx.$$

3.4.2.2. *Discrete orthogonality and discrete $\Xi^{s+}-$transforms*

The finite set of points is given by

$$F_M^{s+} = \frac{1}{M} P^\vee / Q^\vee \cap F^{s+}.$$

We define the corresponding finite set of weights as

$$\Lambda_M^{s+} = P/MQ \cap MF^{s+\vee}$$

where

$$F^{s+\vee} = F^\vee \cup r_s(F^\vee \setminus (Y_s^\vee \cup Y_0^\vee)).$$

Then, for $\lambda, \lambda' \in \Lambda_M^{s+}$, the following discrete orthogonality relations hold:

$$\sum_{x \in F_M^{s+}} \varepsilon^s(x) \Xi_\lambda^{s+}(x)\overline{\Xi_{\lambda'}^{s+}(x)} = kM^2 h_\lambda^{s\vee} \delta_{\lambda\lambda'} \tag{3.17}$$

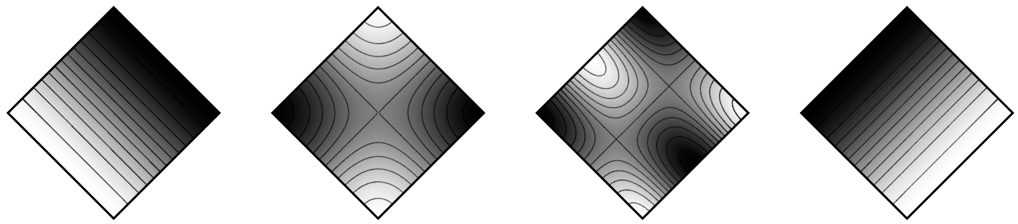

FIG. 3.4. The contour plots of $\Xi^{s+}-$functions of C_2 over the fundamental domain $F^{s+}(C_2)$.

where $h^{s\vee}_\lambda$, k are given by (3.11), (3.12), respectively. The discrete symmetrized $\Xi^{s+}-$function expansion is given by

$$f(x) = \sum_{\lambda\in\Lambda^{s+}_M} c^{s+}_\lambda \Xi^{s+}_\lambda(x), \quad \text{where } c^{s+}_\lambda = \frac{1}{kM^2 h^{s\vee}_\lambda} \sum_{x\in F^{s+}_M} \varepsilon^s(x) f(x) \overline{\Xi^{s+}_\lambda(x)}.$$

3.4.2.3. $\Xi^{s+}-$*functions of* C_2

For a point with coordinates in $\alpha^\vee$-basis (x, y) we have the following explicit form of $\Xi^{s+}-$functions of C_2:

$$\Xi^{s+}_{(a,b)}(x,y) = 2\left\{\cos(2\pi(ax+by)) + \cos(2\pi((a+2b)x - by))\right\}.$$

The fundamental domain F^{s+} is of the form

$$F^{s+}(C_2) = \{x\omega_s^\vee + y\omega_l^\vee \,|\, x,\, y \geq 0, 2x + y \leq 1\}$$
$$\cup \{-x\omega_s^\vee + (2x+y)\omega_l^\vee \,|\, x > 0, y \geq 0, 2x + y \leq 1\}$$

and the lattice of weights P_{s+} is given by

$$P_{s+}(C_2) = \left\{a\omega_s + b\omega_l \,|\, a,\, b \in \mathbb{Z}^{\geq 0}\right\} \cup \left\{-a\omega_s + (a+b)\omega_l \,|\, a \in \mathbb{N}, b \in \mathbb{Z}^{\geq 0}\right\}.$$

The contour plots of some lowest $\Xi^{s+}-$functions of C_2 are given in Figure 3.4. The coefficients d^s_λ of continuous orthogonality relations (3.16) are given in Table 3.1.

The discrete grid F^{s+}_M is given by

$$F^{s+}_M(C_2) = \left\{\frac{a}{M}\omega_s^\vee + \frac{b}{M}\omega_l^\vee \,|\, c,\, a,\, b \in \mathbb{Z}^{\geq 0},\, c + 2a + b = M\right\}$$
$$\cup \left\{-\frac{a}{M}\omega_s^\vee + \frac{2a+b}{M}\omega_l^\vee \,|\, a \in \mathbb{N}, c,\, b \in \mathbb{Z}^{\geq 0},\, c + 2a + b = M\right\}$$

and the corresponding finite set of weights has the form

$$\Lambda^{s+}_M(C_2) = \left\{a\omega_s + b\omega_l \,|\, c,\, a,\, b \in \mathbb{Z}^{\geq 0},\, c + a + 2b = M\right\}$$
$$\cup \left\{-a\omega_s + (a+b)\omega_l \,|\, a,\, c \in \mathbb{N},\, b \in \mathbb{Z}^{\geq 0},\, c + a + 2b = M\right\}.$$

The coefficients $\varepsilon^s(x)$ and $h^{s\vee}_\lambda$ of discrete orthogonality relations (3.17) are given in Table 3.2.

3.4.2.4. Ξ^{s+}*−functions of* G_2

For a point with coordinates in $\alpha^\vee$-basis (x, y) we have the following explicit form of Ξ^{s+}−functions of G_2:

$$\begin{aligned}\Xi^{s+}_{(a,b)}(x,y) =& e^{2\pi i(ax+by)} + e^{2\pi i(-ax+(3a+b)y)} + e^{2\pi i((2a+b)x-(3a+2b)y)} \\ &+ e^{2\pi i((a+b)x-(3a+2b)y)} + e^{2\pi i(-(2a+b)x+(3a+b)y)} + e^{2\pi i(-(a+b)x+by)}.\end{aligned}$$

The fundamental domain F^{s+} is of the form

$$F^{s+}(G_2) = \{x\omega_l^\vee + y\omega_s^\vee \,|\, x,\, y \geq 0, 2x + 3y \leq 1\}$$
$$\cup \{(x+3y)\omega_l^\vee - y\omega_s^\vee \,|\, x \geq 0, y > 0, 2x + 3y \leq 1\}$$

and the lattice of weights P_{s+} is given by

$$P_{s+}(G_2) = \left\{a\omega_l + b\omega_s \,|\, a,\, b \in \mathbb{Z}^{\geq 0}\right\} \cup \left\{(a+b)\omega_l - b\omega_s \,|\, a \in \mathbb{Z}^{\geq 0}, b \in \mathbb{N}\right\}.$$

The contour plots of some lowest Ξ^{s+}−functions of G_2 are given in Figure 3.5. The coefficients d^s_λ of continuous orthogonality relations (3.16) are given in Table 3.1.

The discrete grid F^{s+}_M is given by

$$F^{s+}_M(G_2) = \left\{\frac{a}{M}\omega_l^\vee + \frac{b}{M}\omega_s^\vee \;\middle|\; a,\, b,\, c \in \mathbb{Z}^{\geq 0},\, c + 2a + 3b = M\right\}$$
$$\cup \left\{\frac{a+3b}{M}\omega_l^\vee - \frac{b}{M}\omega_s^\vee \,\middle|\, c,\, a \in \mathbb{Z}^{\geq 0}, b \in \mathbb{N},\, c + 2a + 3b = M\right\}$$

and the corresponding finite set of weights has the form

$$\Lambda^{s+}_M(G_2) = \left\{a\omega_l + b\omega_s \,|\, a,\, b,\, c \in \mathbb{Z}^{\geq 0},\, c + 3a + 2b = M\right\}$$
$$\cup \left\{(a+b)\omega_l - b\omega_s \,|\, a \in \mathbb{Z}^{\geq 0},\, b,\, c \in \mathbb{N},\, c + 3a + 2b = M\right\}.$$

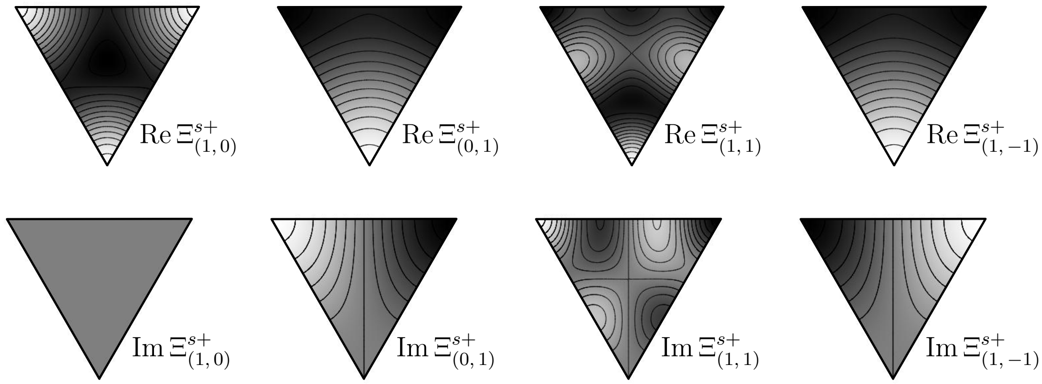

FIG. 3.5. The contour plots of $\Xi^{s+}-$functions of G_2 over the fundamental domain $F^{s+}(G_2)$.

The coefficients $\varepsilon^s(x)$ and $h_\lambda^{s\vee}$ of discrete orthogonality relations (3.17) are given in Table 3.2.

3.4.3. $\Xi^{l+}-$functions

We now focus on the functions $\Psi_\lambda^{\sigma^l}$ which correspond to the sign homomorphism σ^l. We denote these functions by $\Xi_\lambda^{l+} \equiv \Psi_\lambda^{\sigma^l}$ and the corresponding kernel is W^l. The invariance (3.4), (3.15) with respect to $Q^\vee \rtimes W^l$ allows us to consider Ξ_λ^{l+} only on the domain

$$F^{l+} = F \cup r_l(F \setminus (Y_l \cup Y_0)).$$

Similarly, the invariance (3.4) restricts $\lambda \in P$ to the set

$$P_{l+} = P^+ \cup r_l(P^+ \setminus P^l).$$

Thus, we have

$$\Xi_\lambda^{l+}(x) = \sum_{w \in W^l} e^{2\pi i \langle w\lambda, x \rangle}, \quad x \in F^{l+},\ \lambda \in P_{l+}.$$

3.4.3.1. *Continuous orthogonality and $\Xi^{l+}-$transforms*

For any two weights $\lambda, \lambda' \in P_{l+}$ the corresponding $\Xi^{l+}-$functions are orthogonal on F^{l+}

$$\int_{F^{l+}} \Xi_\lambda^{l+}(x)\overline{\Xi_{\lambda'}^{l+}(x)}\,dx = K\,d_\lambda^l\,\delta_{\lambda\lambda'} \tag{3.18}$$

where d_λ^l, K are given by (3.8), (3.9), respectively. The $\Xi^{l+}-$functions determine symmetrized Fourier series expansions,

$$f(x) = \sum_{\lambda\in P_{l+}} c_\lambda^{l+}\Xi_\lambda^{l+}(x), \quad \text{where } c_\lambda^{l+} = \frac{1}{Kd_\lambda^l}\int_{F^{l+}} f(x)\overline{\Xi_\lambda^{l+}(x)}\,dx.$$

3.4.3.2. *Discrete orthogonality and discrete $\Xi^{l+}-$transforms*

The finite set of points is given by

$$F_M^{l+} = \frac{1}{M}P^\vee/Q^\vee \cap F^{l+}.$$

We define the corresponding finite set of weights as

$$\Lambda_M^{l+} = P/MQ \cap MF^{l+\vee}$$

where

$$F^{l+\vee} = F^\vee \cup r_l(F^\vee \setminus Y_l^\vee).$$

Then, for $\lambda, \lambda' \in \Lambda_M^{l+}$, the discrete orthogonality relations hold

$$\sum_{x\in F_M^{l+}} \varepsilon^l(x)\Xi_\lambda^{l+}(x)\overline{\Xi_{\lambda'}^{l+}(x)} = kM^2h_\lambda^{l\vee}\delta_{\lambda\lambda'} \tag{3.19}$$

where $h_\lambda^{l\vee}$, k are given by (3.11), (3.12), respectively. The discrete symmetrized $\Xi^{l+}-$functions expansion is given by

$$f(x) = \sum_{\lambda\in\Lambda_M^{l+}} c_\lambda^{l+}\Xi_\lambda^{l+}(x), \quad \text{where } c_\lambda^{l+} = \frac{1}{kM^2h_\lambda^{l\vee}}\sum_{x\in F_M^{l+}} \varepsilon^l(x)f(x)\overline{\Xi_\lambda^{l+}(x)}.$$

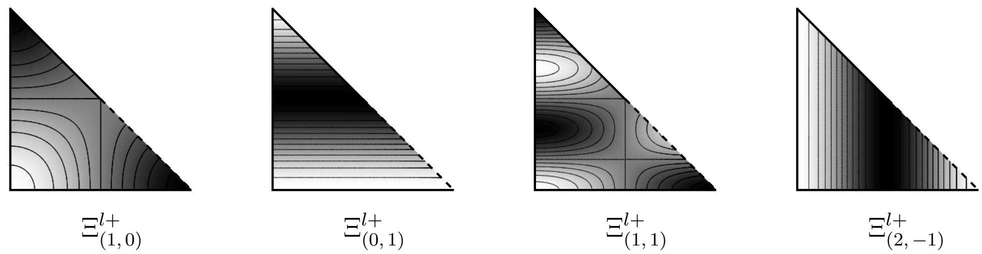

FIG. 3.6. The contour plots of $\Xi^{l+}-$functions of C_2 over the fundamental domain $F^{l+}(C_2)$. The dashed part of the boundary does not belong to the fundamental domain.

3.4.3.3. Ξ^{l+}*−functions of* C_2

For a point with coordinates in $\alpha^\vee$-basis (x, y) we have the following explicit form of $\Xi^{l+}-$functions of C_2:

$$\Xi^{l+}_{(a,b)}(x,y) = 2\left\{\cos(2\pi(ax+by)) + \cos(2\pi(ax-(a+b)y))\right\}.$$

The fundamental domain F^{l+} is of the form

$$\begin{aligned}F^{l+}(C_2) = &\{x\omega_s^\vee + y\omega_l^\vee \,|\, x,\, y \geq 0,\, 2x+y \leq 1\}\\ &\cup \{(x+y)\omega_s^\vee - y\omega_l^\vee \,|\, x \geq 0,\, y > 0,\, 2x+y < 1\}\end{aligned}$$

and the lattice of weights P_{l+} is given by

$$P_{l+}(C_2) = \left\{a\omega_s + b\omega_l \,|\, a,\, b \in \mathbb{Z}^{\geq 0}\right\} \cup \left\{(a+2b)\omega_s - b\omega_l \,|\, a \in \mathbb{Z}^{\geq 0},\, b \in \mathbb{N}\right\}.$$

The contour plots of some lowest $\Xi^{l+}-$functions of C_2 are given in Figure 3.6. The coefficients d^l_λ of continuous orthogonality relations (3.18) are given in Table 3.1.

The discrete grid F^{l+}_M is given by

$$\begin{aligned}F^{l+}_M(C_2) = &\left\{\frac{a}{M}\omega_s^\vee + \frac{b}{M}\omega_l^\vee \,|\, a,\, b,\, c \in \mathbb{Z}^{\geq 0},\, c+2a+b = M\right\}\\ &\cup \left\{\frac{a+b}{M}\omega_s^\vee - \frac{b}{M}\omega_l^\vee \,|\, a \in \mathbb{Z}^{\geq 0},\, b,\, c \in \mathbb{N},\, c+2a+b = M\right\}\end{aligned}$$

and the corresponding finite set of weights has the form

$$\begin{aligned}\Lambda^{l+}_M(C_2) = &\left\{a\omega_s + b\omega_l \,|\, a,\, b,\, c \in \mathbb{Z}^{\geq 0},\, c+a+2b = M\right\}\\ &\cup \left\{(a+2b)\omega_s - b\omega_l \,|\, a,\, c \in \mathbb{Z}^{\geq 0},\, b \in \mathbb{N},\, c+a+2b = M\right\}.\end{aligned}$$

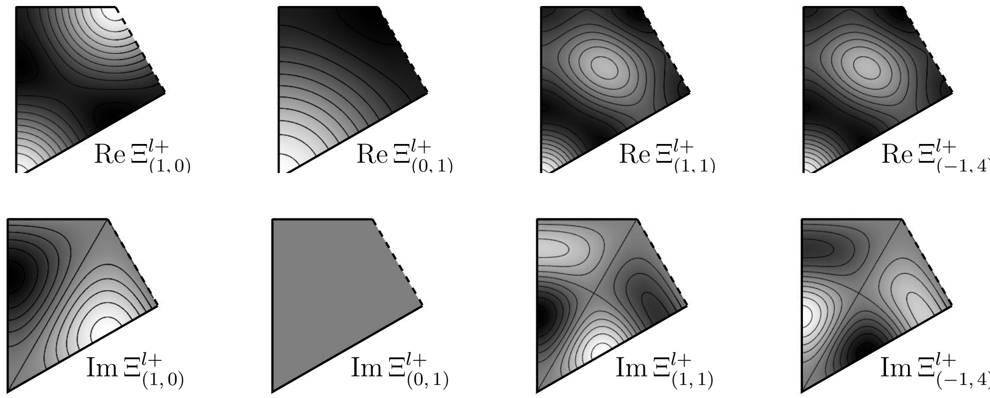

FIG. 3.7. The contour plots of $\Xi^{l+}-$functions of G_2 over the fundamental domain $F^{l+}(G_2)$. The dashed part of the boundary does not belong to the fundamental domain.

The coefficients $\varepsilon^l(x)$ and $h_\lambda^{l\vee}$ of discrete orthogonality relations (3.19) are given in Table 3.2.

3.4.3.4. $\Xi^{l+}-$*functions of* G_2

For a point with coordinates in $\alpha^\vee$-basis (x, y) we have the following explicit form of $\Xi^{l+}-$functions of G_2:

$$\begin{aligned}\Xi^{l+}_{(a,b)}(x,y) =& e^{2\pi i(ax+by)} + e^{2\pi i((a+b)x-by)} + e^{2\pi i(-(2a+b)x+(3a+2b)y)} \\ &+ e^{2\pi i((a+b)x-(3a+2b)y)} + e^{2\pi i(-(2a+b)x+(3a+b)y)} + e^{2\pi i(ax-(3a+b)y)}.\end{aligned}$$

The fundamental domain F^{l+} is of the form

$$\begin{aligned}F^{l+}(G_2) = &\{x\omega_l^\vee + y\omega_s^\vee \,|\, x,\, y \geq 0,\, 2x+3y \leq 1\} \\ &\cup \{-x\omega_l^\vee + (x+y)\omega_s^\vee \,|\, x > 0,\, y \geq 0,\, 2x+3y < 1\}\end{aligned}$$

and the lattice of weights P_{l+} is given by

$$P_{l+}(G_2) = \left\{a\omega_l + b\omega_s \,|\, a,\, b \in \mathbb{Z}^{\geq 0}\right\} \cup \left\{-a\omega_l + (3a+b)\omega_s \,|\, a \in \mathbb{N},\, b \in \mathbb{Z}^{\geq 0}\right\}.$$

The contour plots of some lowest $\Xi^{l+}-$functions of G_2 are given in Figure 3.7. The coefficients d_λ^l of continuous orthogonality relations (3.18) are given in Table 3.1.

The discrete grid F_M^{l+} is given by

$$F_M^{l+}(G_2)=\left\{\frac{a}{M}\omega_l^\vee+\frac{b}{M}\omega_s^\vee\,|\,a,\,b,\,c\in\mathbb{Z}^{\geq 0},\,c+2a+3b=M,\right\}$$
$$\cup\left\{-\frac{a}{M}\omega_l^\vee+\frac{a+b}{M}\omega_s^\vee\,|\,a,\,c\in\mathbb{N},\,b\in\mathbb{Z}^{\geq 0},\,c+2a+3b=M\right\}$$

and the corresponding finite set of weights has the form

$$\Lambda_M^{l+}(G_2)=\left\{a\omega_l+b\omega_s\,|\,a,\,b,\,c\in\mathbb{Z}^{\geq 0},c+3a+2b=M\right\}$$
$$\cup\left\{-a\omega_l+(3a+b)\omega_s\,|\,a\in\mathbb{N},\,b,\,c\in\mathbb{Z}^{\geq 0},\,c+3a+2b=M\right\}.$$

The coefficients $\varepsilon^l(x)$ and $h_\lambda^{l\vee}$ of discrete orthogonality relations (3.19) are given in Table 3.2.

3.5. Mixed even orbit functions

Considering two different homomorphisms $\sigma,\widetilde{\sigma}\neq\imath$ and corresponding kernels $\ker\sigma$, $\ker\widetilde{\sigma}$, we may define another type of 'mixed even' orbit functions $\Psi_\lambda^{\sigma,\widetilde{\sigma}}:\mathbb{R}^2\to\mathbb{C}$ parametrized by $\lambda\in P$, $\sigma\neq\widetilde{\sigma}$

$$\Psi_\lambda^{\sigma,\widetilde{\sigma}}(x)=\sum_{w\in\ker\sigma}\widetilde{\sigma}(w)e^{2\pi i\langle w\lambda,\,x\rangle}. \tag{3.20}$$

These 'mixed even' orbit functions are in general invariant or anti-invariant with respect to the action of $w\in\ker\sigma$

$$\Psi_\lambda^{\sigma,\widetilde{\sigma}}(wx)=\widetilde{\sigma}(w)\Psi_\lambda^{\sigma,\widetilde{\sigma}}(x)$$
$$\Psi_{w\lambda}^{\sigma,\widetilde{\sigma}}(x)=\widetilde{\sigma}(w)\Psi_\lambda^{\sigma,\widetilde{\sigma}}(x)$$

and the invariance with respect to shifts from $Q^\vee$ also holds

$$\Psi_\lambda^{\sigma,\widetilde{\sigma}}(x+q^\vee)=\Psi_\lambda^{\sigma,\widetilde{\sigma}}(x),\quad q^\vee\in Q^\vee. \tag{3.21}$$

We have the following equalities of the mixed even orbit functions which correspond to all possible choices of the sign homomorphisms

$$\Psi_\lambda^{\sigma^e,\sigma^s}=\Psi_\lambda^{\sigma^e,\sigma^l},\quad \Psi_\lambda^{\sigma^l,\sigma^e}=\Psi_\lambda^{\sigma^l,\sigma^s},\quad \Psi_\lambda^{\sigma^s,\sigma^e}=\Psi_\lambda^{\sigma^s,\sigma^l}.$$

Thus, the mixed even orbit functions naturally split into three distinct classes.

3.5.1. Ξ^{e-}−functions

We discuss in detail the functions $\Psi_\lambda^{\sigma^e,\sigma^s} = \Psi_\lambda^{\sigma^e,\sigma^l}$; we denote these functions by Ξ_λ^{e-}, $\lambda \in P$ and the corresponding kernel is W^e. The (anti)invariance (3.5), (3.21) implies that these functions have common zeros in F:

$$\Xi_\lambda^{e-}(x) = 0, \quad x \in (Y_l \cup Y_0) \cap Y_s. \tag{3.22}$$

Taking into account (3.5), (3.21) together with (3.22), we restrict the functions Ξ_λ^{e-} to the domain

$$F^{e-} = (F \setminus \{(Y_l \cup Y_0) \cap Y_s\}) \, \cup \, r_s F^\circ.$$

Similarly, the invariance (3.5) restricts $\lambda \in P$ to the set

$$P_{e-} = (P^+ \setminus (P^l \cap P^s)) \cup r_s P^{++}.$$

Thus, we have

$$\Xi_\lambda^{e-}(x) = \sum_{w \in W^e} \sigma^s(w) e^{2\pi i \langle w\lambda,\, x\rangle}, \quad x \in F^{e-}, \lambda \in P_{e-}.$$

3.5.1.1. *Continuous orthogonality and Ξ^{e-}−transforms*

For any two weights $\lambda, \lambda' \in P_{e-}$ are the corresponding Ξ^{e-}−functions orthogonal on F^{e-}

$$\int_{F^{e-}} \Xi_\lambda^{e-}(x) \overline{\Xi_{\lambda'}^{e-}(x)} \, dx = K \, d_\lambda^e \, \delta_{\lambda\lambda'} \tag{3.23}$$

where d_λ^e, K are given by (3.8), (3.9), respectively. The Ξ^{e-}−functions determine symmetrized Fourier series expansions,

$$f(x) = \sum_{\lambda \in P_{e-}} c_\lambda^{e-} \Xi_\lambda^{e-}(x), \quad \text{where } c_\lambda^{e-} = \frac{1}{K d_\lambda^e} \int_{F^{e-}} f(x) \overline{\Xi_\lambda^{e-}(x)} \, dx.$$

3.5.1.2. *Discrete orthogonality and discrete Ξ^{e-}−transforms*

The finite set of points is given by

$$F_M^{e-} = \frac{1}{M} P^\vee / Q^\vee \cap F^{e-}.$$

We define the corresponding finite set of weights as

$$\Lambda_M^{e-} = P/MQ \cap MF^{e-\vee}$$

where

$$F^{e-\vee} = (F^\vee \setminus \{(Y_s^\vee \cup Y_0^\vee) \cap Y_l^\vee\}) \cup r_s F^{\vee\circ}.$$

Then, for $\lambda, \lambda' \in \Lambda_M^{e-}$,the following discrete orthogonality relations hold

$$\sum_{x \in F_M^{e-}} \varepsilon^e(x) \Xi_\lambda^{e-}(x) \overline{\Xi_{\lambda'}^{e-}(x)} = kM^2 h_\lambda^{e\vee} \delta_{\lambda\lambda'} \tag{3.24}$$

where $h_\lambda^{e\vee}$, k are given by (3.11), (3.12), respectively. The discrete symmetrized $\Xi^{e-}-$functions expansion is given by

$$f(x) = \sum_{\lambda \in \Lambda_M^{e-}} c_\lambda^{e-} \Xi_\lambda^{e-}(x), \quad \text{where } c_\lambda^{e-} = \frac{1}{kM^2 h_\lambda^{e\vee}} \sum_{x \in F_M^{e-}} \varepsilon^e(x) f(x) \overline{\Xi_\lambda^{e-}(x)}.$$

3.5.1.3. $\Xi^{e-}-$*functions of* C_2

For a point with coordinates in $\alpha^\vee$-basis (x, y) we have the following explicit form of $\Xi^{e-}-$functions of C_2:

$$\Xi_{(a,b)}^{e-}(x,y) = 2\left\{\cos(2\pi(ax+by)) - \cos(2\pi((a+2b)x-(a+b)y))\right\}.$$

The fundamental domain F^{e-} is of the form

$$\begin{aligned} F^{e-}(C_2) = &\{x\omega_s^\vee + y\omega_l^\vee \,|\, x,\, y \geq 0, 2x+y \leq 1,\, (x,y) \neq (0,0), (0,1)\} \\ &\cup \{-x\omega_s^\vee + (2x+y)\omega_l^\vee \,|\, x,\, y > 0, 2x+y < 1\} \end{aligned}$$

and the lattice of weights P_{e-} is given by

$$\begin{aligned} P_{e-}(C_2) = &\left\{a\omega_s + b\omega_l \,|\, a,\, b \in \mathbb{Z}^{\geq 0},\, (a,b) \neq (0,0)\right\} \\ &\cup \{-a\omega_s + (a+b)\omega_l \,|\, a,\, b \in \mathbb{N}\}. \end{aligned}$$

The contour plots of some lowest $\Xi^{e-}-$functions of C_2 are given in Figure 3.8. The coefficients d_λ^e of continuous orthogonality relations (3.23) are given in Table 3.1.

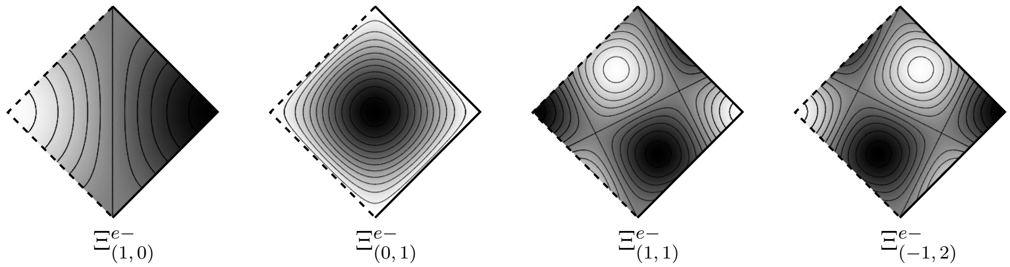

FIG. 3.8. The contour plots of Ξ^{e-}−functions of C_2 over the fundamental domain $F^{e-}(C_2)$. The dashed part of the boundary does not belong to the fundamental domain.

The discrete grid F^{e-}_M is given by

$$\begin{aligned} F^{e-}_M(C_2) = &\left\{ \frac{a}{M}\omega_s^\vee + \frac{b}{M}\omega_l^\vee \,|\, c,\ a,\ b \in \mathbb{Z}^{\geq 0},\ c + 2a + b = M, \right. \\ &\left. [c,a,b] \neq [M,0,0],[0,0,M] \right\} \\ &\cup \left\{ -\frac{a}{M}\omega_s^\vee + \frac{2a+b}{M}\omega_l^\vee \,|\, a,\ b,\ c \in \mathbb{N},\ c + 2a + b = M \right\} \end{aligned}$$

and the corresponding finite set of weights has the form

$$\begin{aligned} \Lambda^{e-}_M(C_2) = &\left\{ a\omega_s + b\omega_l \,|\, c,\ a,\ b \in \mathbb{Z}^{\geq 0},\ c + a + 2b = M, \right. \\ &\left. [c,a,b] \neq [M,0,0],[0,M,0] \right\} \\ &\cup \left\{ -a\omega_s + (a+b)\omega_l \,|\, a,\ b,\ c \in \mathbb{N},\ c + a + 2b = M \right\}. \end{aligned}$$

The coefficients $\varepsilon^e(x)$ and $h^{e\vee}_\lambda$ of discrete orthogonality relations (3.24) are given in Table 3.2.

3.5.1.4. Ξ^{e-}−*functions of* G_2

For a point with coordinates in $\alpha^\vee$-basis (x,y) we have the following explicit form of Ξ^{e-}−functions of G_2:

$$\begin{aligned} \Xi^{e-}_{(a,b)}(x,y) = &2i\{\sin(2\pi(ax+by)) + \sin(2\pi((3a+b)y) - (2a+b)x) \\ &+ \sin(2\pi((a+b)x - (3a+2b)x))\}. \end{aligned}$$

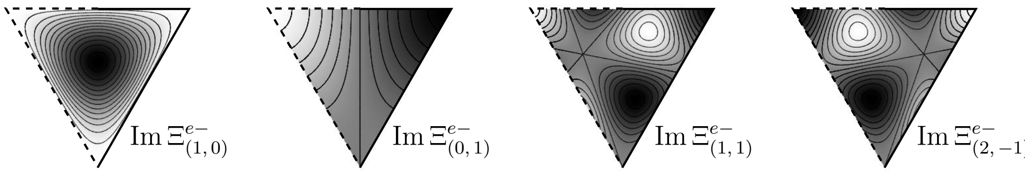

FIG. 3.9. The contour plots of Ξ^{e-}−functions of G_2 over the fundamental domain $F^{e-}(G_2)$. The dashed part of the boundary does not belong to the fundamental domain. Real parts of Ξ^{e-}-functions are zero.

The fundamental domain F^{e-} is of the form

$$\begin{aligned}F^{e-}(G_2) = &\{x\omega_l^\vee + y\omega_s^\vee \,|\, x,\, y \geq 0, 2x+3y \leq 1,\, (x,y) \neq (0,0), (1/2,0)\}\\ &\cup \{(x+3y)\omega_l^\vee - y\omega_s^\vee \,|\, x,\, y > 0,\, 2x+3y < 1\}\end{aligned}$$

and the lattice of weights P_{e-} is given by

$$P_{e-}(G_2) = \big\{a\omega_l + b\omega_s \,|\, a,\, b \in \mathbb{Z}^{\geq 0},\, (a,b) \neq (0,0)\big\} \cup \{(a+b)\omega_l - b\omega_s \,|\, a,\, b \in \mathbb{N}\}$$

The contour plots of some lowest Ξ^{e-}−functions of G_2 are given in Figure 3.9.

The coefficients d^e_λ of continuous orthogonality relations (3.23) are given in Table 3.1. The discrete grid F^{e-}_M is given by

$$\begin{aligned}F^{e-}_M(G_2) = &\left\{\frac{a}{M}\omega_l^\vee + \frac{b}{M}\omega_s^\vee \;|\; a,\, b,\, c \in \mathbb{Z}^{\geq 0},\, c+2a+3b = M,\right.\\ &(c,a,b) \neq (M,0,0),\, (0,M/2,0)\}\\ &\cup \left\{\frac{a+3b}{M}\omega_l^\vee - \frac{b}{M}\omega_s^\vee \,|\, c,\, a,\, b \in \mathbb{N},\, c+2a+3b = M\right\}\end{aligned}$$

and the corresponding finite set of weights has the form

$$\begin{aligned}\Lambda^{e-}_M(G_2) = &\big\{a\omega_l + b\omega_s \,|\, a,\, b,\, c \in \mathbb{Z}^{\geq 0},\, c+3a+2b = M,\, [c,a,b] \neq [M,0,0],\\ &[0,0,M/2]\big\} \cup \{(a+b)\omega_l - b\omega_s \,|\, a,\, b,\, c \in \mathbb{N},\, c+3a+2b = M\}\,.\end{aligned}$$

The coefficients $\varepsilon^e(x)$ and $h^{e\vee}_\lambda$ of discrete orthogonality relations (3.24) are given in Table 3.2.

3.5.2. Ξ^{s-}–functions

We discuss in detail the functions $\Psi_\lambda^{\sigma^s,\sigma^e} = \Psi_\lambda^{\sigma^s,\sigma^l}$; we denote these functions by Ξ_λ^{s-}, $\lambda \in P$ and the corresponding kernel is W^s. The (anti)invariance (3.5), (3.21) implies that these functions have common zeros in F:

$$\Xi_\lambda^{s-}(x) = 0, \quad x \in Y_l \cup Y_0. \tag{3.25}$$

Taking into account (3.5), (3.21) together with (3.25), we restrict the functions Ξ_λ^{s-} to the domain

$$F^{s-} = (F \setminus (Y_l \cup Y_0)) \cup r_s F^\circ.$$

Similarly, the invariance (3.5) restricts $\lambda \in P$ to the set

$$P_{s-} = (P^+ \setminus P_l) \cup r_s P^{++}.$$

Thus, we have

$$\Xi_\lambda^{s-}(x) = \sum_{w \in W^s} \sigma^l(w) e^{2\pi i \langle w\lambda, x\rangle}, \quad x \in F^{s-}, \lambda \in P_{s-}.$$

3.5.2.1. *Continuous orthogonality and Ξ^{s-}–transforms*

For any two weights $\lambda, \lambda' \in P_{s-}$ the corresponding Ξ^{s-}–functions are orthogonal on F^{s-}

$$\int_{F^{s-}} \Xi_\lambda^{s-}(x)\overline{\Xi_{\lambda'}^{s-}(x)}\, dx = K\, \delta_{\lambda\lambda'} \tag{3.26}$$

where K is given by (3.9). The Ξ^{s-}–functions determine symmetrized Fourier series expansions,

$$f(x) = \sum_{\lambda \in P_{s-}} c_\lambda^{s-} \Xi_\lambda^{s-}(x), \quad \text{where } c_\lambda^{s-} = \frac{1}{K}\int_{F^{s-}} f(x)\overline{\Xi_\lambda^{s-}(x)}\, dx.$$

3.5.2.2. *Discrete orthogonality and discrete Ξ^{s-}–transforms*

The finite set of points is given by

$$F_M^{s-} = \frac{1}{M} P^\vee / Q^\vee \cap F^{s-}.$$

We define the corresponding finite set of weights as

$$\Lambda_M^{s-} = P/MQ \cap MF^{s-\vee}$$

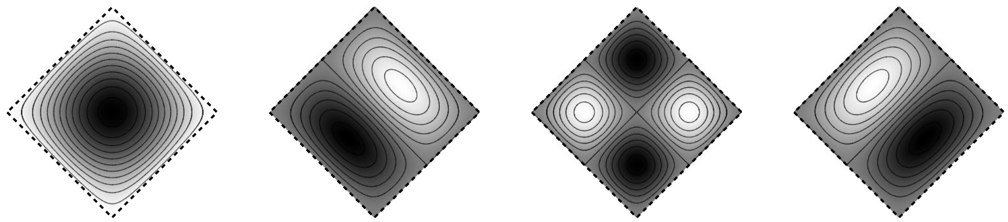

FIG. 3.10. The contour plots of Ξ^{s-}−functions of C_2 over the fundamental domain $F^{s-}(C_2)$. The dashed part of the boundary does not belong to the fundamental domain.

where

$$F^{s-\vee} = (F^\vee \setminus Y_l^\vee) \cup r_s F^{\vee\circ}.$$

Then, for $\lambda, \lambda' \in \Lambda_M^{s-}$, the following discrete orthogonality relations hold

$$\sum_{x\in F_M^{s-}} \varepsilon^s(x)\Xi_\lambda^{s-}(x)\overline{\Xi_{\lambda'}^{s-}(x)} = kM^2 h_\lambda^{s\vee}\delta_{\lambda\lambda'} \tag{3.27}$$

where $h_\lambda^{s\vee}$, k are given by (3.11), (3.12), respectively. The discrete symmetrized Ξ^{s-}−functions expansion is given by

$$f(x) = \sum_{\lambda\in\Lambda_M^{s-}} c_\lambda^{s-}\Xi_\lambda^{s-}(x), \quad \text{where } c_\lambda^{s-} = \frac{1}{kM^2h_\lambda^{s\vee}} \sum_{x\in F_M^{s-}} \varepsilon^s(x) f(x)\overline{\Xi_\lambda^{s-}(x)}.$$

3.5.2.3. Ξ^{s-}−*functions of* C_2

For a point with coordinates in $\alpha^\vee$-basis (x, y) we have the following explicit form of Ξ^{s-}−functions of C_2:

$$\Xi^{s-}_{(a,b)}(x,y) = 2\left\{\cos(2\pi(ax+by)) - \cos(2\pi((a+2b)x - by))\right\}.$$

The fundamental domain F^{s-} is of the form

$$\begin{aligned} F^{s-}(C_2) = &\{x\omega_s^\vee + y\omega_l^\vee \,|\, x\geq 0,\, y>0,\, 2x+y<1\} \\ &\cup \{-x\omega_s^\vee + (2x+y)\omega_l^\vee \,|\, x,\, y>0,\, 2x+y<1\} \end{aligned}$$

and the lattice of weights P_{s-} is given by

$$P_{s-}(C_2) = \left\{a\omega_s + b\omega_l \,|\, a\in\mathbb{Z}^{\geq 0},\, b\in\mathbb{N}\right\} \cup \{-a\omega_s + (a+b)\omega_l \,|\, a,\, b\in\mathbb{N}\}.$$

The contour plots of some lowest Ξ^{s-}−functions of C_2 are given in Figure 3.10.

The discrete grid F_M^{s-} is given by

$$F_M^{s-}(C_2) = \left\{ \frac{a}{M}\omega_s^\vee + \frac{b}{M}\omega_l^\vee \,|\, a \in \mathbb{Z}^{\geq 0},\, b,\, c \in \mathbb{N},\, c + 2a + b = M \right\} \cup \left\{ -\frac{a}{M}\omega_s^\vee + \frac{2a+b}{M}\omega_l^\vee \,|\, a,\, b,\, c \in \mathbb{N},\, c + 2a + b = M \right\}$$

and the corresponding finite set of weights has the form

$$\Lambda_M^{s-}(C_2) = \left\{ a\omega_s + b\omega_l \,|\, c,\, a \in \mathbb{Z}^{\geq 0},\, b \in \mathbb{N},\, c + a + 2b = M \right\} \cup \left\{ -a\omega_s + (a+b)\omega_l \,|\, a,\, b,\, c \in \mathbb{N},\, c + a + 2b = M \right\}.$$

The coefficients $\varepsilon^s(x)$ of discrete orthogonality relations (3.27) have a common value $\varepsilon^s(x) = 4$ for all $x \in F_M^{s-}(C_2)$. The coefficients $h_\lambda^{s\vee}$ are given in Table 3.2.

3.5.2.4. Ξ^{s-}*−functions of* G_2

For a point with coordinates in $\alpha^\vee$-basis (x, y) we have the following explicit form of Ξ^{s-}−functions of G_2:

$$\begin{aligned}\Xi_{(a,b)}^{s-}(x,y) =& e^{2\pi i(ax+by)} - e^{2\pi i(-ax+(3a+b)y)} - e^{2\pi i((2a+b)x-(3a+2b)y)} \\ &+ e^{2\pi i((a+b)x-(3a+2b)y)} + e^{2\pi i(-(2a+b)x+(3a+b)y)} - e^{2\pi i(-(a+b)x+by)}.\end{aligned}$$

The fundamental domain F^{s-} is of the form

$$F^{s-}(G_2) = \{ x\omega_l^\vee + y\omega_s^\vee \,|\, x > 0,\, y \geq 0, 2x + 3y < 1 \} \cup \{ (x+3y)\omega_l^\vee - y\omega_s^\vee \,|\, x,\, y > 0,\, 2x + 3y < 1 \}$$

and the lattice of weights P_{s-} is given by

$$P_{s-}(G_2) = \left\{ a\omega_l + b\omega_s \,|\, a \in \mathbb{N},\, b \in \mathbb{Z}^{\geq 0} \right\} \cup \{ (a+b)\omega_l - b\omega_s \,|\, a,\, b \in \mathbb{N} \}.$$

The contour plots of some lowest Ξ^{s-}−functions of G_2 are given in Figure 3.11.

The discrete grid F_M^{s-} is given by

$$F_M^{s-}(G_2) = \left\{ \frac{a}{M}\omega_l^\vee + \frac{b}{M}\omega_s^\vee \;|\; b \in \mathbb{Z}^{\geq 0},\, a,\, c \in \mathbb{N},\, c + 2a + 3b = M \right\} \cup \left\{ \frac{a+3b}{M}\omega_l^\vee - \frac{b}{M}\omega_s^\vee \,|\, a,\, b,\, c \in \mathbb{N},\, c + 2a + 3b = M \right\}$$

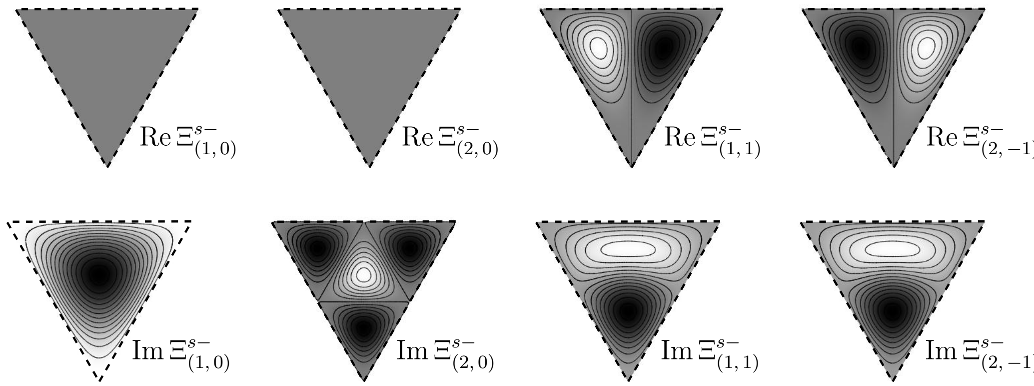

FIG. 3.11. The contour plots of Ξ^{s-}−functions of G_2 over the fundamental domain $F^{s-}(G_2)$. The dashed part of the boundary does not belong to the fundamental domain.

and the corresponding finite set of weights has the form

$$\Lambda^{s-}_M(G_2) = \left\{ a\omega_l + b\omega_s \,|\, a \in \mathbb{N},\, b,\, c \in \mathbb{Z}^{\geq 0},\, c + 3a + 2b = M \right\}$$
$$\cup \left\{ (a+b)\omega_l - b\omega_s \,|\, a,\, b,\, c \in \mathbb{N},\, c + 3a + 2b = M \right\}.$$

The coefficients $\varepsilon^s(x)$ of discrete orthogonality relations (3.27) have a common value $\varepsilon^s(x) = 6$ for all $x \in F^{s-}_M(G_2)$. The coefficients $h^{s\vee}_\lambda$ are given in Table 3.2.

3.5.3. Ξ^{l-}−functions

We discuss in detail the functions $\Psi^{\sigma^l,\sigma^e}_\lambda = \Psi^{\sigma^l,\sigma^s}_\lambda$; we denote these functions by Ξ^{l-}_λ, $\lambda \in P$ and the corresponding kernel is W^l. The (anti)invariance (3.5), (3.21) implies that these functions have common zeros in F:

$$\Xi^{l-}_\lambda(x) = 0, \quad x \in Y_s. \tag{3.28}$$

Taking into account (3.5), (3.21) together with (3.28), we restrict the functions Ξ^{l-}_λ to the domain

$$F^{l-} = (F \setminus Y_s) \cup r_l F^\circ.$$

Similarly, the invariance (3.5) restricts $\lambda \in P$ to the set

$$P_{l-} = (P^+ \setminus P_s) \cup r_l P^{++}.$$

Thus, we have

$$\Xi_\lambda^{l-}(x) = \sum_{w\in W^l} \sigma^s(w) e^{2\pi i\langle w\lambda,\, x\rangle}, \quad x \in F^{l-},\, \lambda \in P_{l-}.$$

3.5.3.1. *Continuous orthogonality and $\Xi^{l-}-$transforms*

For any two weights $\lambda, \lambda' \in P_{l-}$ the corresponding $\Xi^{l-}-$functions are orthogonal on F^{l-}

$$\int_{F^{l-}} \Xi_\lambda^{l-}(x)\overline{\Xi_{\lambda'}^{l-}(x)}\, dx = K\, \delta_{\lambda\lambda'} \tag{3.29}$$

where K is given by (3.9). The $\Xi^{l-}-$functions determine symmetrized Fourier series expansions,

$$f(x) = \sum_{\lambda\in P_{l-}} c_\lambda^{l-} \Xi_\lambda^{l-}(x), \quad \text{where } c_\lambda^{l-} = \frac{1}{K}\int_{F^{l-}} f(x)\overline{\Xi_\lambda^{l-}(x)}\, dx.$$

3.5.3.2. *Discrete orthogonality and discrete $\Xi^{l-}-$transforms*

The finite set of points is given by

$$F_M^{l-} = \frac{1}{M} P^\vee / Q^\vee \cap F^{l-}.$$

We define the corresponding finite set of weights as

$$\Lambda_M^{l-} = P/MQ \cap MF^{l-\vee}$$

where

$$F^{l-\vee} = (F^\vee \setminus (Y_s^\vee \cup Y_0^\vee)) \cup r_l F^{\vee\circ}.$$

Then, for $\lambda, \lambda' \in \Lambda_M^{l-}$, the following discrete orthogonality relations hold

$$\sum_{x\in F_M^{l-}} \varepsilon^l(x) \Xi_\lambda^{l-}(x)\overline{\Xi_{\lambda'}^{l-}(x)} = kM^2 \delta_{\lambda\lambda'} \tag{3.30}$$

where k is given by (3.12). The discrete symmetrized $\Xi^{l-}-$functions expansion is given by

$$f(x) = \sum_{\lambda\in\Lambda_M^{l-}} c_\lambda^{l-} \Xi_\lambda^{l-}(x), \quad \text{where } c_\lambda^{l-} = \frac{1}{kM^2}\sum_{x\in F_M^{l-}} \varepsilon^l(x) f(x)\overline{\Xi_\lambda^{l-}(x)}.$$

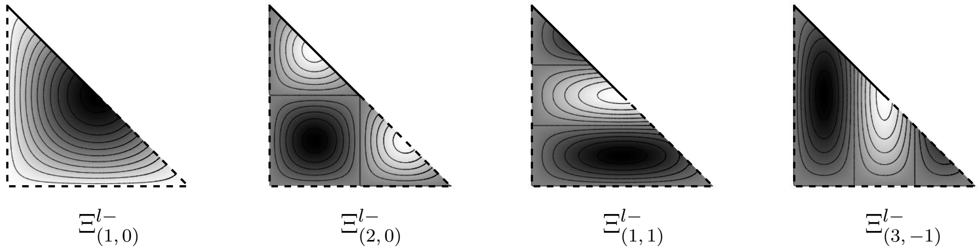

FIG. 3.12. The contour plots of Ξ^{l-}–functions of C_2 over the fundamental domain $F^{l-}(C_2)$. The dashed part of the boundary does not belong to the fundamental domain.

3.5.3.3. Ξ^{l-}*–functions of* C_2

For a point with coordinates in $\alpha^\vee$-basis (x,y) we have the following explicit form of Ξ^{l-}–functions of C_2:

$$\Xi^{l-}_{(a,b)}(x,y) = 2\left\{\cos(2\pi(ax+by)) - \cos(2\pi(ax-(a+b)y))\right\}.$$

The fundamental domain F^{l-} is of the form

$$\begin{aligned} F^{l-}(C_2) &= \{x\omega_s^\vee + y\omega_l^\vee \,|\, x>0,\, y\geq 0,\, 2x+y\leq 1\} \\ &\cup \{(x+y)\omega_s^\vee - y\omega_l^\vee \,|\, x,\, y>0,\, 2x+y<1\} \end{aligned}$$

and the lattice of weights P_{l-} is given by

$$P_{l-}(C_2) = \left\{a\omega_s + b\omega_l \,|\, a\in\mathbb{N},\, b\in\mathbb{Z}^{\geq 0}\right\} \cup \left\{(a+2b)\omega_s - b\omega_l \,|\, a,\, b\in\mathbb{N}\right\}.$$

The contour plots of some lowest Ξ^{l-}–functions of C_2 are given in Figure 3.12.

The discrete grid F^{l-}_M is given by

$$\begin{aligned} F^{l-}_M(C_2) &= \left\{\frac{a}{M}\omega_s^\vee + \frac{b}{M}\omega_l^\vee \,|\, a\in\mathbb{N},\, b,\, c\in\mathbb{Z}^{\geq 0},\, c+2a+b=M\right\} \\ &\cup \left\{\frac{a+b}{M}\omega_s^\vee - \frac{b}{M}\omega_l^\vee \,|\, a,\, b,\, c\in\mathbb{N},\, c+2a+b=M\right\} \end{aligned}$$

and the corresponding finite set of weights has the form

$$\begin{aligned} \Lambda^{l-}_M(C_2) &= \left\{a\omega_s + b\omega_l \,|\, a,\, c\in\mathbb{N},\, b\in\mathbb{Z}^{\geq 0},\, c+a+2b=M\right\} \\ &\cup \{(a+2b)\omega_s - b\omega_l \,|\, a,\, b,\, c\in\mathbb{N},\, c+a+2b=M\}. \end{aligned}$$

The coefficients $\varepsilon^l(x)$ of discrete orthogonality relations (3.30) are given in Table 3.2.

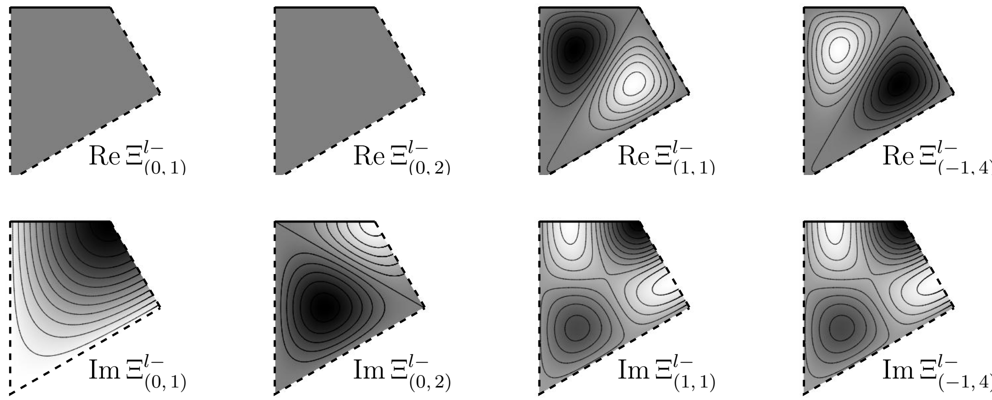

FIG. 3.13. The contour plots of $\Xi^{l-}-$functions of G_2 over the fundamental domain $F^{l-}(G_2)$. The dashed part of the boundary does not belong to the fundamental domain.

3.5.3.4. Ξ^{l-}*−functions of* G_2

For a point with coordinates in $\alpha^\vee$-basis (x, y) we have the following explicit form of $\Xi^{l-}-$functions of G_2:

$$\begin{aligned}\Xi^{l-}_{(a,b)}(x,y) =& e^{2\pi i(ax+by)} - e^{2\pi i((a+b)x-by)} - e^{2\pi i(-(2a+b)x+(3a+2b)y)} \\ &+ e^{2\pi i((a+b)x-(3a+2b)y)} + e^{2\pi i(-(2a+b)x+(3a+b)y)} - e^{2\pi i(ax-(3a+b)y)}.\end{aligned}$$

The fundamental domain F^{l-} is of the form

$$\begin{aligned}F^{l-}(G_2) = &\{x\omega_l^\vee + y\omega_s^\vee \,|\, x \geq 0,\ y > 0,\ 2x+3y \leq 1\} \\ &\cup \{-x\omega_l^\vee + (x+y)\omega_s^\vee \,|\, x,\ y > 0,\ 2x+3y < 1\}\end{aligned}$$

and the lattice of weights P_{l-} is given by

$$P_{l-}(G_2) = \left\{a\omega_l + b\omega_s \,|\, a \in \mathbb{Z}^{\geq 0},\ b \in \mathbb{N}\right\} \cup \{-a\omega_l + (3a+b)\omega_s \,|\, a,\ b \in \mathbb{N}\}.$$

The contour plots of some lowest $\Xi^{l-}-$functions of G_2 are given in Figure 3.13.

The discrete grid F^{l-}_M is given by

$$\begin{aligned}F^{l-}_M(G_2) = &\left\{\frac{a}{M}\omega_l^\vee + \frac{b}{M}\omega_s^\vee \,|\, a,\ c \in \mathbb{Z}^{\geq 0},\ b \in \mathbb{N},\ c+2a+3b = M,\right\} \\ &\cup \left\{-\frac{a}{M}\omega_l^\vee + \frac{a+b}{M}\omega_s^\vee \,|\, a,\ b,\ c \in \mathbb{N},\ c+2a+3b = M\right\}\end{aligned}$$

and the corresponding finite set of weights has the form

$$\Lambda^{l-}_M(G_2) = \left\{ a\omega_l + b\omega_s \,|\, a \in \mathbb{Z}^{\geq 0},\, b,\, c \in \mathbb{N},\, c + 3a + 2b = M \right\}$$
$$\cup \left\{ -a\omega_l + (3a+b)\omega_s \,|\, a,\, b,\, c \in \mathbb{N},\, c + 3a + 2b = M \right\}.$$

The coefficients $\varepsilon^l(x)$ of discrete orthogonality relations (3.30) are given in Table 3.2.

3.6. Product decompositions

Various products of generalized $E-$functions can be decomposed into the sum of $E-$functions. We distinguish the following three cases.

3.6.1. $\Xi^{e\pm} \cdot \Xi^{e\pm}$

The product of two $\Xi^{e+}-$ or two $\Xi^{e-}-$functions decomposes into the sum of Ξ^{e+} functions, the signs of the summands are all positive in the first case. We have the following general formulas of these decompositions which hold for any $\lambda, \lambda' \in P$ and $x \in \mathbb{R}^2$

$$\Xi^{e+}_\lambda(x) \cdot \Xi^{e+}_{\lambda'}(x) = \sum_{w \in W^e} \Xi^{e+}_{\lambda + w\lambda'}(x), \quad \Xi^{e-}_\lambda(x) \cdot \Xi^{e-}_{\lambda'}(x) = \sum_{w \in W^e} \sigma^s(w) \Xi^{e+}_{\lambda + w\lambda'}(x). \tag{3.31}$$

The mixed product of $\Xi^{e+}-$ and $\Xi^{e-}-$functions decomposes into the sum of $\Xi^{e-}-$functions:

$$\Xi^{e+}_\lambda(x) \cdot \Xi^{e-}_{\lambda'}(x) = \sum_{w \in W^e} \sigma^s(w) \Xi^{e-}_{\lambda + w\lambda'}(x). \tag{3.32}$$

Example 3.6.1. Using the explicit form of the even orbits of C_2 in Figure 3.2, we demonstrate the decompositions (3.31), (3.32). In the following formulas we choose $\lambda = (5,3)$, $\lambda' = (1,1)$, omit the variables (x,y) of the $\Xi^{e+}-$ and $\Xi^{e-}-$functions of C_2 and write explicitly the products:

$$\Xi^{e+}_{(5,3)} \cdot \Xi^{e+}_{(1,1)} = \Xi^{e+}_{(6,4)} + \Xi^{e+}_{(2,5)} + \Xi^{e+}_{(8,1)} + \Xi^{e+}_{(4,2)}$$
$$\Xi^{e-}_{(5,3)} \cdot \Xi^{e-}_{(1,1)} = \Xi^{e+}_{(6,4)} - \Xi^{e+}_{(2,5)} - \Xi^{e+}_{(8,1)} + \Xi^{e+}_{(4,2)}$$
$$\Xi^{e+}_{(5,3)} \cdot \Xi^{e-}_{(1,1)} = \Xi^{e-}_{(6,4)} - \Xi^{e-}_{(2,5)} - \Xi^{e-}_{(8,1)} + \Xi^{e-}_{(4,2)}.$$

3.6.2. $\Xi^{s\pm}\cdot\Xi^{s\pm}$

The product of two $\Xi^{s+}-$ or two $\Xi^{s-}-$functions decomposes into the sum of Ξ^{s+} functions, the signs of the summands are all positive in the first case. We have the following general formulas of these decompositions which hold for any $\lambda,\lambda'\in P$ and $x\in\mathbb{R}^2$

$$\Xi^{s+}_{\lambda}(x)\cdot\Xi^{s+}_{\lambda'}(x)=\sum_{w\in W^s}\Xi^{s+}_{\lambda+w\lambda'}(x),\quad \Xi^{s-}_{\lambda}(x)\cdot\Xi^{s-}_{\lambda'}(x)=\sum_{w\in W^s}\sigma^l(w)\Xi^{s+}_{\lambda+w\lambda'}(x). \tag{3.33}$$

The mixed product of $\Xi^{s+}-$ and $\Xi^{s-}-$functions decomposes into the sum of $\Xi^{s-}-$functions:

$$\Xi^{s+}_{\lambda}(x)\cdot\Xi^{s-}_{\lambda'}(x)=\sum_{w\in W^s}\sigma^l(w)\Xi^{s-}_{\lambda+w\lambda'}(x). \tag{3.34}$$

Example 3.6.2. Using the explicit form of the even orbits of C_2 in Figure 3.2, we demonstrate the decompositions (3.33), (3.34). In the following formulas we choose $\lambda=(5,3)$, $\lambda'=(1,1)$, omit the variables (x,y) of the $\Xi^{s+}-$ and $\Xi^{s-}-$functions of C_2 and write explicitly the products:

$$\begin{aligned}
\Xi^{s+}_{(5,3)}\cdot\Xi^{s+}_{(1,1)}&=\Xi^{s+}_{(6,4)}+\Xi^{s+}_{(2,4)}+\Xi^{s+}_{(8,2)}+\Xi^{s+}_{(4,2)}\\
\Xi^{s-}_{(5,3)}\cdot\Xi^{s-}_{(1,1)}&=\Xi^{s+}_{(6,4)}-\Xi^{s+}_{(2,4)}-\Xi^{s+}_{(8,2)}+\Xi^{s+}_{(4,2)}\\
\Xi^{s+}_{(5,3)}\cdot\Xi^{s-}_{(1,1)}&=\Xi^{s-}_{(6,4)}-\Xi^{s-}_{(2,4)}-\Xi^{s-}_{(8,2)}+\Xi^{s-}_{(4,2)}.
\end{aligned}$$

3.6.3. $\Xi^{l\pm}\cdot\Xi^{l\pm}$

The product of two $\Xi^{l+}-$ or two $\Xi^{l-}-$functions decomposes into the sum of Ξ^{l+} functions, the signs of the summands are all positive in the first case. We have the following general formulas of these decompositions which hold for any $\lambda,\lambda'\in P$ and $x\in\mathbb{R}^2$

$$\Xi^{l+}_{\lambda}(x)\cdot\Xi^{l+}_{\lambda'}(x)=\sum_{w\in W^l}\Xi^{l+}_{\lambda+w\lambda'}(x),\quad \Xi^{l-}_{\lambda}(x)\cdot\Xi^{l-}_{\lambda'}(x)=\sum_{w\in W^l}\sigma^s(w)\Xi^{l+}_{\lambda+w\lambda'}(x). \tag{3.35}$$

The mixed product of $\Xi^{l+}-$ and $\Xi^{l-}-$functions decomposes into the sum of $\Xi^{l-}-$functions:

$$\Xi^{l+}_{\lambda}(x) \cdot \Xi^{l-}_{\lambda'}(x) = \sum_{w \in W^l} \sigma^s(w) \Xi^{l-}_{\lambda + w\lambda'}(x). \tag{3.36}$$

Example 3.6.3. Using the explicit form of the even orbits of C_2 in Figure 3.2, we demonstrate the decompositions (3.35), (3.36). In the following formulas we choose $\lambda = (5,3)$, $\lambda' = (1,1)$, omit the variables (x,y) of the $\Xi^{l+}-$ and $\Xi^{l-}-$functions of C_2 and write explicitly the products:

$$\begin{aligned}
\Xi^{l+}_{(5,3)} \cdot \Xi^{l+}_{(1,1)} &= \Xi^{l+}_{(6,4)} + \Xi^{l+}_{(6,1)} + \Xi^{l+}_{(4,5)} + \Xi^{l+}_{(4,2)} \\
\Xi^{l-}_{(5,3)} \cdot \Xi^{l-}_{(1,1)} &= \Xi^{l+}_{(6,4)} - \Xi^{l+}_{(2,4)} - \Xi^{l+}_{(4,5)} + \Xi^{l+}_{(4,2)} \\
\Xi^{l+}_{(5,3)} \cdot \Xi^{l-}_{(1,1)} &= \Xi^{l-}_{(6,4)} - \Xi^{l-}_{(6,1)} - \Xi^{l-}_{(4,5)} + \Xi^{l-}_{(4,2)}.
\end{aligned}$$

3.7. Concluding Remarks

- The short roots $W\Delta^s$ of the Weyl group of C_2 (G_2) is of the type $A_1 \times A_1$ (A_2). This leads to the fact, that in the case of C_2 the functions Ξ^{s+} can be identified with C-orbit functions of the group of $A_1 \times A_1$ and Ξ^{s-} with the S-orbit functions of $A_1 \times A_1$. In particular, the function $\Xi^{s+}_{(a,b)}$ of C_2 differs only by rotation by $\pi/4$ from the C-function $\varphi_{a+b,b}$ of $A_1 \times A_1$ (and similarly for the Ξ^{s-}). In the case of G_2, the functions $\Xi^{s+}_{(a,b)}$ differ from the functions $\varphi_{(a,a+b)}$ of A_2 by the rotation by $\pi/3$ (and similarly for the Ξ^{s-}).

- For all groups (3.1) there are precisely six families of E-function analogous to the rank 2 cases here. The kernels of defining homomorphisms are the symmetry groups of the subsystems of roots of the same length. Namely,

$$W\Delta^s \text{ is of type } \begin{cases} nA_1 \text{ in } B_n \\ D_n \text{ in } C_n \\ D_4 \text{ in } F_4 \end{cases}, \tag{3.37}$$

$$W\Delta^l \text{ is of type } \begin{cases} D_n \text{ in } B_n \\ nA_1 \text{ in } C_n \\ D_4 \text{ in } F_4 \end{cases},$$

where nA_1 denotes the semisimple Lie algebra, with $A_1 \times \cdots \times A_1$ multiplied n times. In (3.37) we use the known isomorphisms $D_2 = A_1 \times A_1$, $D_3 = A_3$, and $B_2 = C_2$.

- The most frequently analyzed 2-dimensional digital data are sampled on rectangular domains. Such data need first to be placed in F_M with appropriate choice of the integer M to match the density of the data lattice and the lattice points in F_M. That is done more efficiently when F_M matches more closely the shape of the data lattice. The functions on Figures 4, 8, and 10 illustrate the new choices one did not have so far.

- For now unexplored remain the properties of the E-functions of all the types when the dominant weight of the function is a point in $\mathbb{R}^n$ but not a point of the weight lattice $P \subset \mathbb{R}^n$. Definitions and many properties of such functions are analogs of the properties of the E-functions described here. Obvious application of such functions are Fourier integral expansions as opposed to Fourier series.

- Weyl group orbit functions are closely related to multivariable orthogonal polynomials. That relation exists also in the case of E-functions of all the types. In our opinion this relation merits an explicit description.

- Very little is known about arithmetic properties of the E-functions, except of those facts that can be readily deduced from the rich arithmetic properties of the characters [**40**],[**47**].

Acknowledgements

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Chapter 4

FOUR FAMILIES OF WEYL GROUP ORBIT FUNCTIONS OF B_3 AND OF C_3.

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Abstract: The properties of the four families of special functions of three real variables, called here $C-$, $S-$, S^s- and S^l-functions, are studied. The S^s- and S^l-functions are considered in all details required for their exploitation in Fourier expansions of digital data, sampled on finite fragment of lattices of any density and of the $3D$ symmetry imposed by the weight lattices of B_3 and C_3 simple Lie algebras/groups. The continuous interpolations, which are induced by the discrete expansions, are exemplified and compared for some model functions.

4.1. Introduction

Four families of functions, depending on three real variables, called here as $C-$, $S-$, S^s- and S^l-functions, are described. Each family is complete within its functional space [**37**] and orthogonal when integrated over a finite region F of real Euclidean space $\mathbb{R}^3$. Moreover, the functions of each family are discretely orthogonal on a fraction of a lattice in F of density of our choice and of symmetry dictated by the simple Lie algebras B_3 and C_3.

The definition of these functions for any number of variables and some of their properties are described in [**24, 25, 37**]. The first two families of functions are also called $C-$ and $S-$functions. They are well known as constituents of irreducible characters of compact simple Lie groups. Indeed, the irreducible characters can be expressed as finite sums of $C-$functions with integer coefficients called dominant weight multiplicities. Ratio of two $S-$functions appears in the Weyl character formula [**20**]. Two other families are called S^l- and S^s- functions. Inspired by the Weyl character formula, we can consider so called hybrid characters, i.e. the ratio of two S^l- or S^s- functions. It is possible to show that they decompose into the sums of $C-$functions with integer coefficients. The explicit formulas for coefficients are in [**32**].

It is well known that the $C-$functions play an important role in the definition of Jacobi polynomials [**14, 27, 28**]. One can also remark that the characters and the hybrid characters could be derived as special cases of Jacobi polynomials. However, important insight into the properties of these functions would have been lost in the generality of the approach. For example, their discrete orthogonality appears to be outside of that approach, which is a handicap in the world of ever growing amount of digital data. From our perspective particularly useful are the four families discretized within F. The problem of discretization of the $C-$ and $S-$functions, which is now over 20 years old [**18, 41, 48, 45**], is carried over to the dicretization of the other two families in [**17**].

Our motivation for studying these families of functions is guided by ever increasing need to process three-dimensional digital data. Discrete orthogonality of these functions open new efficient possibilities for precisely that. Moreover, the $3D$ symmetry imposed by the weight lattices of B_3 and C_3 should be advantageous in describing quantum systems possessing such a symmetry, as well as in some problems of quantum information theory.

There are two undoubtedly interesting extensions of this work. The first one is to combine pairs of functions from the present four families into so called $E-$functions, two-variable generalization of the common exponential function are found in [**2**]. In this way we can find six families of functions, which again are orthogonal as continuous and also as discrete functions over finite extension of F.

The second possible extension of this work is to three-variables orthogonal polynomials. Curiously, the extensive literature about the polynomials contains little information about their discretization. This work opens a way to study these problems.

The paper is organized as follows. Section 2 reviews some basic notations concerning the root systems and Weyl groups. We present the short and long fundamental domains of the affine Weyl group of B_3 and C_3. In the section 3 we define four families of orbit functions. In the section 4 we describe in detail orbit functions S^s- and S^l- including their discrete orthogonality and discrete transforms.

4.2. Root systems, affine Weyl groups and fundamental domains

4.2.1. Root systems

Consider a simple Lie algebra of rank three and its ordered set of simple roots $\Delta = (\alpha_1, \alpha_2, \alpha_3)$. The set Δ forms a basis of the three-dimensional real Euclidean space $\mathbb{R}^3$ **[20, 22]** and satisfies certain specific conditions. There are only two simple Lie algebras of rank three which, with respect to the standard scalar product $\langle \cdot, \cdot \rangle$, have two different lenghts of their simple roots — B_3 and C_3. The set Δ is for these two cases decomposed into the set of short simple roots Δ_s and the set of long simple roots Δ_l:

$$\Delta = \Delta_s \cup \Delta_l. \tag{4.1}$$

The set Δ is usually described by the Coxeter–Dynkin diagram and its corresponding Coxeter matrix M or, equivalently, by the Cartan matrix C. The vectors called coroots $\alpha_i^\vee$ are defined as renormalizations of roots: $\alpha_i^\vee = 2\alpha_i / \langle \alpha_i, \alpha_i \rangle$, $i = 1, 2, 3$. In addition to the $\alpha-$basis of simple roots and the $\alpha^\vee-$basis, the following two bases are useful: the weight $\omega-$basis, defined by the relations

$$\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}\,, \quad i, j \in \{1, 2, 3\},$$

and the coweight $\omega^\vee-$basis, given by renormalization as $\omega_i^\vee = 2\omega_i / \langle \alpha_i, \alpha_i \rangle$, $i = 1, 2, 3$.

Standardly, the root lattice Q is the set of all integer linear combinations of the simple roots

$$Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$$

and the coroot $Q^\vee$ lattice is

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee + \mathbb{Z}\alpha_3^\vee.$$

The weight lattice and the coweight lattice are given standardly as

$$P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3, \quad P^\vee = \mathbb{Z}\omega_1^\vee + \mathbb{Z}\omega_2^\vee + \mathbb{Z}\omega_3^\vee.$$

Two important subsets of the weight lattice P are the cone of dominant weights P^+ and the cone of strictly dominant weights P^{++}:

$$P^+ = \mathbb{Z}^{\geq 0}\omega_1 + \mathbb{Z}^{\geq 0}\omega_2 + \mathbb{Z}^{\geq 0}\omega_3 \;\supset\; P^{++} = \mathbb{N}\omega_1 + \mathbb{N}\omega_2 + \mathbb{N}\omega_3.$$

The decomposition (4.1) induces two subsets of P^+ which are crucial for description of the orbit functions. The first excludes points from P^+ which are orthogonal to short roots,

$$P^{+s} = \left\{\omega \in P^+ \,|\, (\forall \alpha \in \Delta_s)(\langle \omega,\, \alpha\rangle > 0)\right\} \tag{4.2}$$

and the second excludes points orthogonal to long roots

$$P^{+l} = \left\{\omega \in P^+ \,|\, (\forall \alpha \in \Delta_l)(\langle \omega,\, \alpha\rangle > 0)\right\}. \tag{4.3}$$

4.2.2. Affine Weyl groups

The reflection r_α, $\alpha \in \Delta$, which fixes the plane orthogonal to α and passing through the origin, is explicitly written for $x \in \mathbb{R}^3$ as $r_\alpha x = x - \langle \alpha, x\rangle \alpha^\vee$. The Weyl group W is a finite group generated by reflections $r_i \equiv r_{\alpha_i}$, $i = 1, 2, 3$. The system of vectors obtained by the action of W on the set of simple roots Δ forms a root system $W\Delta$ which contains its unique highest root $\xi \in W\Delta$. The marks are the coefficients m_1, m_2, m_3 of the highest root ξ in $\alpha-$basis, $\xi = m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3$.

The affine reflection r_0 with respect to this highest root is given by

$$r_0 x = r_\xi x + \frac{2\xi}{\langle \xi,\, \xi\rangle}\,, \qquad r_\xi x = x - \frac{2\langle x,\, \xi\rangle}{\langle \xi,\, \xi\rangle}\xi\,, \qquad x \in \mathbb{R}^3\,.$$

The affine Weyl group W^{aff} is generated by reflections from the set

$$R = \{r_0, r_1, r_2, r_3.\}$$

The decomposition (4.1) induces a decomposition of the generator set R,

$$R = R^s \cup R^l \tag{4.4}$$

where the subsets R^s and R^l are given by

$$\begin{aligned} R^s &= \{r_\alpha \,|\, r_\alpha \in \Delta_s\} \\ R^l &= \{r_\alpha \,|\, r_\alpha \in \Delta_l\} \cup \{r_0\}. \end{aligned}$$

The affine Weyl group W^{aff} consists of orthogonal transformations from W and of shifts by vectors from the coroot lattice $Q^\vee$. The fundamental domain F of the action of W^{aff} on $\mathbb{R}^3$ is a tetrahedron with vertices $\left\{0, \frac{\omega_1^\vee}{m_1}, \frac{\omega_2^\vee}{m_2}, \frac{\omega_3^\vee}{m_3}\right\}$.

The set of reflections corresponding to dual roots $\Delta^\vee = (\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee)$ also generates the Weyl group W. The system of vectors $W\Delta^\vee$ is a root system and contains the highest dual root $\eta \in W\Delta^\vee$. The dual marks are the coefficients $m_1^\vee, m_2^\vee, m_3^\vee$ of the dual highest root η in $\alpha^\vee-$basis, $\eta = m_1^\vee\alpha_1^\vee + m_2^\vee\alpha_2^\vee + m_3^\vee\alpha_3^\vee$.

The dual affine reflection $r_0^\vee$ with respect to the highest dual root is given by

$$r_0^\vee x = r_\eta x + \frac{2\eta}{\langle \eta,\, \eta\rangle}\,, \qquad r_\eta x = x - \frac{2\langle x,\, \eta\rangle}{\langle \eta,\, \eta\rangle}\eta\,, \qquad x \in \mathbb{R}^3\,.$$

The dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ is generated by reflections from the set $R^\vee = \{r_0^\vee,\, r_1,\, r_2,\, r_3\}$, see [**18**]. The decomposition (4.1) also induces a decomposition of the generator set $R^\vee$,

$$R^\vee = \{r_0^\vee, r_1, r_2, r_3\} = R^{s\vee} \cup R^{l\vee} \tag{4.5}$$

where the subsets $R^{s\vee}$ and $R^{l\vee}$ are given by

$$\begin{aligned} R^{s\vee} &= \{r_\alpha \,|\, r_\alpha \in \Delta_s\} \cup \{r_0^\vee\} \\ R^{l\vee} &= \{r_\alpha \,|\, r_\alpha \in \Delta_l\}\,. \end{aligned}$$

The dual affine group $\widehat{W}^{\mathrm{aff}}$ consists of orthogonal transformations from W and of shifts by vectors from the root lattice Q. The dual fundamental domain $F^\vee$ of the action of $\widehat{W}^{\mathrm{aff}}$ on $\mathbb{R}^3$ is a tetrahedron with vertices $\left\{0, \frac{\omega_1}{m_1^\vee}, \frac{\omega_2}{m_2^\vee}, \frac{\omega_3}{m_3^\vee}\right\}$.

4.2.3. Short and long fundamental domains

The boundary of the fundamental domain F consists of points stabilized by the generators from R. Two types of the boundaries of F are determined by the decomposition (4.4) — those points which are stabilized by R^s are collected in the short boundary H^s,

$$H^s = \{a \in F \,|\, (\exists r \in R^s)(ra = a)\}$$

and the points stabilized by R^l in the long boundary H^l,

$$H^l = \left\{a \in F \,|\, (\exists r \in R^l)(ra = a)\right\}.$$

The points from F which do not lie on the short boundary form the short fundamental domain F^s,

$$F^s = F \setminus H^s$$

and the points which do not lie on the long boundary form the long fundamental domain F^l,

$$F^l = F \setminus H^l.$$

Similarly, the boundary of the dual fundamental domain $F^\vee$ consists of points stabilized by the generators from $R^\vee$ and two types of the boundaries of $F^\vee$ are determined by the decomposition (4.5). The points which are stabilized by $R^{s\vee}$ are collected in the short dual boundary $H^{s\vee}$,

$$H^{s\vee} = \{a \in F^\vee \,|\, (\exists r \in R^{s\vee})(ra = a)\}$$

and the points stabilized by $R^{l\vee}$ in the long dual boundary $H^{l\vee}$,

$$H^{l\vee} = \left\{a \in F^\vee \,|\, (\exists r \in R^{l\vee})(ra = a)\right\}.$$

The points from $F^\vee$ which do not lie on the short dual boundary form the short dual fundamental domain $F^{s\vee}$,

$$F^{s\vee} = F^\vee \setminus H^{s\vee}$$

and the points not on the long dal boundary form the long dual fundamental domain $F^{l\vee}$,

$$F^{l\vee} = F^\vee \setminus H^{l\vee}.$$

4.2.4. The Lie algebra B_3

For practical purposes, the most convenient way of specifying the root system Δ and the bases $\alpha^\vee$, $\omega^\vee$ and ω is to evaluate their coordinates in a fixed orthonormal basis. With respect to the standard orthonormal basis of $\mathbb{R}^3$, these four bases of B_3 are of the form

$$
\begin{aligned}
\alpha_1 &= (1,-1,0), & \omega_1 &= (1,0,0), & \alpha_1^\vee &= (1,-1,0), & \omega_1^\vee &= (1,0,0),\\
\alpha_2 &= (0,1,-1), & \omega_2 &= (1,1,0), & \alpha_2^\vee &= (0,1,-1), & \omega_2^\vee &= (1,1,0),\\
\alpha_3 &= (0,0,1), & \omega_3 &= (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}), & \alpha_3^\vee &= (0,0,2), & \omega_3^\vee &= (1,1,1).
\end{aligned}
$$

In this setting it holds that $\langle\alpha_1,\,\alpha_1\rangle = \langle\alpha_2,\,\alpha_2\rangle = 2$ and $\langle\alpha_3,\,\alpha_3\rangle = 1$, which means that α_1, α_2 are the long roots and α_3 is the short root of B_3 — the decomposition (4.1) is

$$\Delta_s = \{\alpha_3\}, \qquad \Delta_l = \{\alpha_1,\alpha_2\}.$$

The short and long subsets (4.2), (4.3) of the grid P^+ are of the form

$$P^{+s} = \mathbb{Z}^{\geq 0}\omega_1 + \mathbb{Z}^{\geq 0}\omega_2 + \mathbb{N}\omega_3, \qquad P^{+l} = \mathbb{N}\omega_1 + \mathbb{N}\omega_2 + \mathbb{Z}^{\geq 0}\omega_3.$$

The highest root ξ and the dual highest root η are given as

$$\xi = \alpha_1 + 2\alpha_2 + 2\alpha_3, \qquad \eta = 2\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$$

which determine the fundamental domain F and the dual fundamental domain $F^\vee$ explicitly as

$$
\begin{aligned}
F &= \left\{y_1\omega_1^\vee + y_2\omega_2^\vee + y_3\omega_3^\vee \,|\, y_0,\, y_1,\, y_2,\, y_3 \in \mathbb{R}^{\geq 0},\, y_0 + y_1 + 2y_2 + 2y_3 = 1\right\},\\
F^\vee &= \left\{z_1\omega_1 + z_2\omega_2 + z_3\omega_3 \,|\, z_0,\, z_1,\, z_2,\, z_3 \in \mathbb{R}^{\geq 0},\, z_0 + 2z_1 + 2z_2 + z_3 = 1\right\}.
\end{aligned}
$$

The induced decompositions (4.4), (4.5) are of the form

$$
\begin{aligned}
R^s &= \{r_3\}, & R^l &= \{r_0,\, r_1,\, r_2\},\\
R^{s\vee} &= \{r_0^\vee,\, r_3\}, & R^{l\vee} &= \{r_1,\, r_2\},
\end{aligned}
$$

which give the short and the long fundamental domains explicitly

$$F^s = \left\{ y_1^s\omega_1^\vee + y_2^s\omega_2^\vee + y_3^s\omega_3^\vee \,|\, y_0^s,\, y_1^s,\, y_2^s \in \mathbb{R}^{\geq 0},\, y_3^s \in \mathbb{R}^{>0},\, y_0^s + y_1^s + 2y_2^s + 2y_3^s = 1 \right\},$$
$$F^l = \left\{ y_1^l\omega_1^\vee + y_2^l\omega_2^\vee + y_3^l\omega_3^\vee \,|\, y_0^l,\, y_1^l,\, y_2^l \in \mathbb{R}^{>0},\, y_3^l \in \mathbb{R}^{\geq 0},\, y_0^l + y_1^l + 2y_2^l + 2y_3^l = 1 \right\},$$

together with their dual versions

$$F^{s\vee} = \left\{ z_1^s\omega_1 + z_2^s\omega_2 + z_3^s\omega_3 \,|\, z_0^s,\, z_3^s \in \mathbb{R}^{>0},\, z_1^s,\, z_2^s \in \mathbb{R}^{\geq 0},\, z_0^s + 2z_1^s + 2z_2^s + z_3^s = 1 \right\},$$
$$F^{l\vee} = \left\{ z_1^l\omega_1 + z_2^l\omega_2 + z_3^l\omega_3 \,|\, z_0^l,\, z_3^l \in \mathbb{R}^{\geq 0},\, z_1^l,\, z_2^l \in \mathbb{R}^{>0},\, z_0^l + 2z_1^l + 2z_2^l + z_3^l = 1 \right\}.$$

The α and $\omega-$bases, together with the fundamental domains F, F^s and F^l of B_3 are depicted in Figure 4.1.

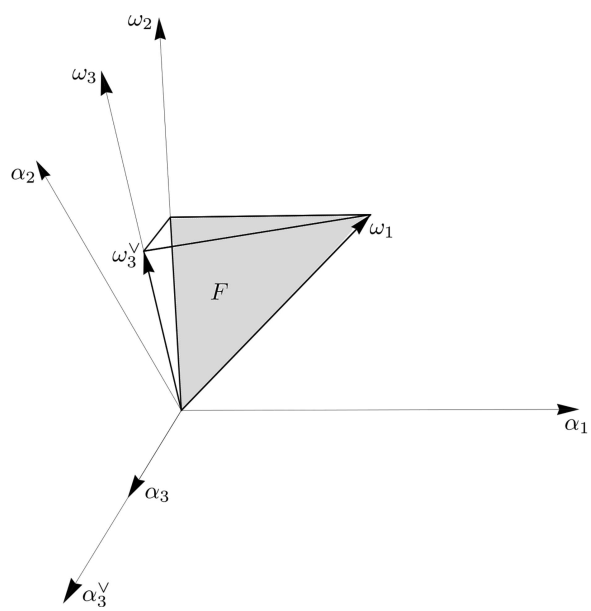

FIG. 4.1. The α and $\omega-$bases and the fundamental domain F of B_3. The tetrahedron F without the grey back face, which depicts H^s, is the short fundamental domain F^s; the tetrahedron F without the three unmarked faces is the long fundamental domain F^l.

4.2.5. The Lie algebra C_3

With respect to the standard orthonormal basis of $\mathbb{R}^3$, the four bases of C_3 are of the form

$$\begin{aligned}
&\alpha_1 = \frac{1}{\sqrt{2}}(1,-1,0), &&\omega_1 = \frac{1}{\sqrt{2}}(1,0,0), &&\alpha_1^\vee = \sqrt{2}\,(1,-1,0), &&\omega_1^\vee = (\sqrt{2},0,0),\\
&\alpha_2 = \frac{1}{\sqrt{2}}(0,1,-1), &&\omega_2 = \frac{1}{\sqrt{2}}(1,1,0), &&\alpha_2^\vee = \sqrt{2}\,(0,1,-1), &&\omega_2^\vee = \sqrt{2}\,(1,1,0),\\
&\alpha_3 = (0,0,\sqrt{2}), &&\omega_3 = \frac{1}{\sqrt{2}}(1,1,1), &&\alpha_3^\vee = (0,0,\sqrt{2}), &&\omega_3^\vee = \frac{1}{\sqrt{2}}(1,1,1).
\end{aligned}$$

In this setting it holds that $\langle\alpha_1,\,\alpha_1\rangle = \langle\alpha_2,\,\alpha_2\rangle = 1$ and $\langle\alpha_3,\,\alpha_3\rangle = 2$, which means that $\alpha_1,\ \alpha_2$ are the short roots and α_3 is the long root of C_3 — the decomposition (4.1) is

$$\Delta_s = \{\alpha_1, \alpha_2\}, \qquad \Delta_l = \{\alpha_3\}.$$

The short and long subsets (4.2), (4.3) of the grid P^+ are of the form

$$P^{+s} = \mathbb{N}\omega_1 + \mathbb{N}\omega_2 + \mathbb{Z}^{\geq 0}\omega_3, \qquad P^{+l} = \mathbb{Z}^{\geq 0}\omega_1 + \mathbb{Z}^{\geq 0}\omega_2 + \mathbb{N}\omega_3.$$

The highest root ξ and the dual highest root η are given as

$$\xi = 2\alpha_1 + 2\alpha_2 + \alpha_3, \qquad \eta = \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee$$

which determine the fundamental domain F and the dual fundamental domain $F^\vee$ explicitly as

$$\begin{aligned}
F &= \left\{ y_1\omega_1^\vee + y_2\omega_2^\vee + y_3\omega_3^\vee \,|\, y_0,\ y_1,\ y_2,\ y_3 \in \mathbb{R}^{\geq 0},\ y_0 + 2y_1 + 2y_2 + y_3 = 1 \right\},\\
F^\vee &= \left\{ z_1\omega_1 + z_2\omega_2 + z_3\omega_3 \,|\, z_0,\ z_1,\ z_2,\ z_3 \in \mathbb{R}^{\geq 0},\ z_0 + z_1 + 2z_2 + 2z_3 = 1 \right\}.
\end{aligned}$$

The induced decompositions (4.4), (4.5) are of the form

$$\begin{aligned}
R^s &= \{r_1,\ r_2\}, &\qquad R^l &= \{r_0,\ r_3\},\\
R^{s\vee} &= \{r_0^\vee,\ r_1,\ r_2\}, &\qquad R^{l\vee} &= \{r_3\},
\end{aligned}$$

which give the short and the long fundamental domains explicitly

$$\begin{aligned}
F^s &= \left\{ y_1^s\omega_1^\vee + y_2^s\omega_2^\vee + y_3^s\omega_3^\vee \,|\, y_0^s,\ y_3^s \in \mathbb{R}^{\geq 0},\ y_1^s,\ y_2^s \in \mathbb{R}^{>0},\ y_0^s + 2y_1^s + 2y_2^s + y_3^s = 1 \right\},\\
F^l &= \left\{ y_1^l\omega_1^\vee + y_2^l\omega_2^\vee + y_3^l\omega_3^\vee \,|\, y_0^l,\ y_3^l \in \mathbb{R}^{>0},\ y_1^l,\ y_2^l \in \mathbb{R}^{\geq 0},\ y_0^l + 2y_1^l + 2y_2^l + y_3^l = 1 \right\},
\end{aligned}$$

together with their dual versions

$$F^{s\vee} = \left\{ z_1^s\omega_1 + z_2^s\omega_2 + z_3^s\omega_3 \,|\, z_0^s,\, z_1^s,\, z_2^s \in \mathbb{R}^{>0},\, z_3^s \in \mathbb{R}^{\geq 0},\, z_0^s + z_1^s + 2z_2^s + 2z_3^s = 1 \right\},$$

$$F^{l\vee} = \left\{ z_1^l\omega_1 + z_2^l\omega_2 + z_3^l\omega_3 \,|\, z_0^l,\, z_1^l,\, z_2^l \in \mathbb{R}^{\geq 0},\, z_3^l \in \mathbb{R}^{>0},\, z_0^l + z_1^l + 2z_2^l + 2z_3^l = 1 \right\}.$$

The α and $\omega-$bases, together with the fundamental domains F, F^s and F^l of C_3 are depicted in Figure 4.2.

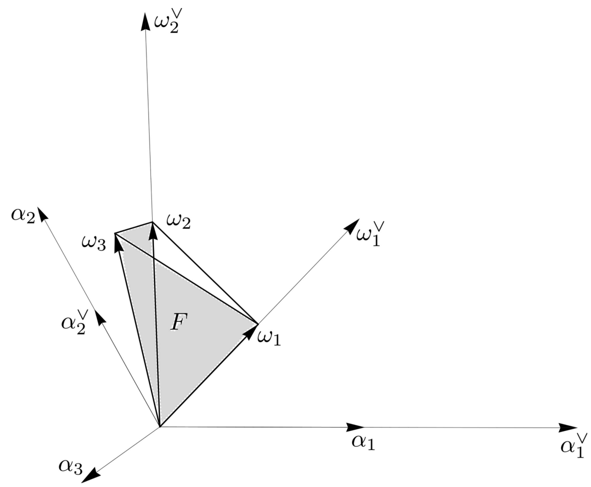

FIG. 4.2. The α and $\omega-$bases and the fundamental domain F of C_3. The tetrahedron F without the two grey faces, which depict H^s, is the short fundamental domain F^s; the tetrahedron F without the two unmarked faces is the long fundamental domain F^l.

4.3. Orbit functions

4.3.1. Orbits and stabilizers

Considering any $\lambda \in \mathbb{R}^3$, the stabilizer $\mathrm{Stab}_W(\lambda)$ of λ is the set $\mathrm{Stab}_W(\lambda) = \{w \in W \,|\, w\lambda = \lambda\}$ and its order is denoted by d_λ,

$$d_\lambda \equiv |\mathrm{Stab}_W(\lambda)|. \tag{4.6}$$

For calculation of continuous orthogonality of various types of orbit functions, the number of elements in the Weyl group W and the volume of the fundamental domain F are needed. Their product $|W||F|$ is denoted by K and it holds that (see e.g. [**18**])

$$K \equiv |W||F| = \begin{cases} 2 & \text{for } B_3 \\ 2\sqrt{2} & \text{for } C_3. \end{cases} \tag{4.7}$$

The orbits and the stabilizers on the torus $\mathbb{R}^3/Q^\vee$ are needed for the discrete calculus of orbit functions. An arbitrarily chosen natural number M controls the density of the grids appearing in this calculus [**18**]. The discrete calculus of orbit functions is performed over the finite group $\frac{1}{M}P^\vee/Q^\vee$. The finite complement set of weights is taken as the quotient group P/MQ. For any $x \in \mathbb{R}^3/Q^\vee$, its orbit by the action of W is given by $Wx = \{wx \in \mathbb{R}^3/Q^\vee \mid w \in W\}$ and its order is denoted by $\varepsilon(x)$,

$$\varepsilon(x) \equiv |Wx|. \tag{4.8}$$

For any $\lambda \in P/MQ$, its stabilizer $\mathrm{Stab}^\vee(\lambda)$ by the action of W is given by

$$\mathrm{Stab}^\vee(\lambda) = \{w \in W \mid w\lambda = \lambda\}\,.$$

and its order is denoted by $h^\vee_\lambda$

$$h^\vee_\lambda \equiv |\mathrm{Stab}^\vee(\lambda)|. \tag{4.9}$$

Moreover, for calculation of discrete orthogonality of various types of orbit functions, the determinant of the Cartan matrix $\det C$ is needed. The product $|W|\det C$ is denoted by k and it holds that (see e.g. [**18**])

$$k \equiv |W|\det C = 96, \qquad \text{for } B_3,\, C_3. \tag{4.10}$$

4.3.2. Four types of orbit functions

The Weyl group W can also be abstractly defined by the presentation of a Coxeter group

$$r_i^2 = 1, \quad (r_i r_j)^{m_{ij}} = 1, \quad i, j = 1,\, 2,\, 3, \tag{4.11}$$

where integers m_{ij} denote elements of the Coxeter matrix. The Coxeter matrices M of B_3 and C_3 are of the form

$$M(B_3) = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}, \qquad M(C_3) = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}. \tag{4.12}$$

Crucial tool for defining the orbit functions are 'sign' homomorphisms $\sigma : W \to \{\pm 1\}$. A sign homomorphism can be defined by prescribing its values on the generators r_1, r_2, r_3 of W. An admissible mapping has to satisfy the presentation condition (4.11)

$$\sigma(r_i)^2 = 1, \quad (\sigma(r_i)\sigma(r_j))^{m_{ij}} = 1, \quad i,j = 1,2,3. \tag{4.13}$$

Two obvious choices $\sigma(r_i) = 1$ and $\sigma^e(r_i) = -1$ for every $i \in \{1,2,3\}$ lead to the standard homomorphisms $\mathbf{1}$ and σ^e with values given for any $w \in W$ as

$$\begin{aligned} \mathbf{1}(w) &= 1, \\ \sigma^e(w) &= \det w. \end{aligned}$$

It turns out that for root systems with two different lengths of roots there are two other available choices [**37**]. This can also be directly seen for the cases of B_3 and C_3 — the non-diagonal elements m_{ij} of the Coxeter matrices (4.12) are even except for the elements corresponding simultaneously to two short roots or two long roots. Therefore, if we set one value of σ on all the short roots and, independently, another value for all the long roots, the admissibility condition (4.13) is still satisfied. Consequently, there are two more sign homomorphisms, denoted by σ^s and σ^l,

$$\sigma^s(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_l \\ -1, & \alpha \in \Delta_s \end{cases}$$

$$\sigma^l(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_s \\ -1, & \alpha \in \Delta_l. \end{cases}$$

Each of the sign homomorphisms $\mathbf{1}$, σ^e, σ^s and σ^l induces a family of complex orbit functions. The functions in each family are labeled by the weights $\lambda \in P$

and their general form is

$$\psi_\lambda^\sigma(a) = \sum_{w\in W} \sigma(w)\, e^{2\pi i\langle w\lambda,\, a\rangle}, \quad a \in \mathbb{R}^3. \tag{4.14}$$

Among the basic general properties of all these families of orbit functions are their invariance with respect to shifts from $q^\vee \in Q^\vee$

$$\psi_\lambda^\sigma(a+q^\vee) = \psi_\lambda^\sigma(a) \tag{4.15}$$

and their invariance or antiinvariance with respect to action of elements from $w \in W$

$$\psi_\lambda^\sigma(wa) = \sigma(w)\, \psi_\lambda^\sigma(a) \tag{4.16}$$

$$\psi_{w\lambda}^\sigma(a) = \sigma(w)\, \psi_\lambda^\sigma(a). \tag{4.17}$$

4.3.3. $C-$ and $S-$functions

Choosing $\sigma = \mathbf{1}$, so-called $C-$functions [**24**] are obtained from (4.14). Following [**18**], these functions are here denoted by the symbol $\Phi_\lambda \equiv \psi_\lambda^{\mathbf{1}}$. The invariance (4.15), (4.16) with respect to the affine Weyl group W^{aff} allows to consider Φ_λ only on the fundamental domain F. Similarly, the invariance (4.17) restricts the weights $\lambda \in P$ to the set P^+. Thus, we have

$$\Phi_\lambda(x) = \sum_{w\in W} e^{2\pi i\langle w\lambda,\, x\rangle}, \qquad x \in F,\ \lambda \in P^+.$$

For the detailed review of $C-$functions see [**24**] and references therein. Continuous and discrete orthogonality of $C-$functions Φ_λ are studied for all rank two cases in detail in [**54, 55**]. The discretization properties of $C-$functions on a finite fragment of the grid $\frac{1}{M}P^\vee$ is described in full generality in [**18**].

Choosing $\sigma = \sigma^e$, well–known $S-$functions [**25**] are obtained from (4.14). Following [**18, 25**], these functions are here denoted by the symbol $\varphi_\lambda \equiv \psi_\lambda^{\sigma^e}$. The invariance (4.15) together with the antiinvariance (4.16) allows to consider φ_λ only on the interior F° of the fundamental domain F. Similarly, the antiinvariance

(4.17) restricts the weights $\lambda \in P$ to the set P^{++}. Thus, we have

$$\varphi_\lambda(x) = \sum_{w\in W} (\det w)\, e^{2\pi i \langle w\lambda,\, x\rangle}, \qquad x \in F^\circ,\ \lambda \in P^{++}.$$

For the detailed review of $S-$functions see [**25**] and references therein. Continuous and discrete orthogonality of $S-$functions φ_λ are studied for all rank two cases in detail in [**56**]. The discretization properties of $S-$functions on a finite fragment of the grid $\frac{1}{M}P^\vee$ is described in full generality in [**18**].

4.4. S^s- AND S^l-FUNCTIONS

4.4.1. S^s-functions

Choosing $\sigma = \sigma^s$, so–called S^s-functions [**37**] are obtained from (4.14). Following [**17**], these functions are here denoted by the symbol $\varphi^s_\lambda \equiv \psi^{\sigma^s}_\lambda$. The antiinvariance (4.16) with respect to the short reflections r_α, $\alpha \in \Delta_s$ together with shift invariance (4.15) imply zero values of S^s-functions on the boundary H^s,

$$\varphi^s_\lambda(a') = 0, \quad a' \in H^s.$$

Therefore, the functions φ^s_λ are considered on the fundamental domain $F^s = F \setminus H^s$ only. Similarly, the antiinvariance (4.17) restricts the weights $\lambda \in P$ to the set P^{+s}. Thus, we have

$$\varphi^s_\lambda(x) = \sum_{w\in W} \sigma^s(w)\, e^{2\pi i \langle w\lambda,\, x\rangle}, \qquad x \in F^s,\ \lambda \in P^{+s}.$$

4.4.1.1. *Continuous orthogonality and S^s-transforms*

For any two weights $\lambda, \lambda' \in P^{+s}$ the corresponding S^s-functions are orthogonal on F^s

$$\int_{F^s} \varphi^s_\lambda(x)\overline{\varphi^s_{\lambda'}(x)}\,\mathrm{d}x = K\, d_\lambda\, \delta_{\lambda\lambda'} \tag{4.18}$$

where d_λ, K are given by (4.6), (4.7), respectively. The S^s-functions determine symmetrized Fourier series expansions,

$$f(x) = \sum_{\lambda\in P^{+s}} c^s_\lambda \varphi^s_\lambda(x), \quad \text{where } c^s_\lambda = \frac{1}{K d_\lambda}\int_{F^s} f(x)\overline{\varphi^s_\lambda(x)}\,\mathrm{d}x.$$

$\lambda \in P^+$	d_λ	$\lambda \in P^+$	d_λ
(a,b,c)	1	$(a,b,0)$	2
$(a,0,c)$	2	$(0,b,c)$	2
$(a,0,0)$	8	$(0,b,0)$	4
$(0,0,c)$	6	$(0,0,0)$	48

TAB. 4.1. Orders of stabilizers of $\lambda \in P^+$ for B_3 and C_3, The coordinates (a,b,c) are in $\omega-$basis with $a,b,c \neq 0$.

4.4.1.2. *Discrete orthogonality and discrete S^s-transforms*

The finite set of points is given by

$$F_M^s = \frac{1}{M} P^\vee / Q^\vee \cap F^s$$

and the corresponding finite set of weights as

$$\Lambda_M^s = P/MQ \cap MF^{s\vee}.$$

Then, for $\lambda, \lambda' \in \Lambda_M^s$, the following discrete orthogonality relations hold,

$$\sum_{x \in F_M^s} \varepsilon(x) \varphi_\lambda^s(x) \overline{\varphi_{\lambda'}^s(x)} = kM^3 h_\lambda^\vee \delta_{\lambda\lambda'} \tag{4.19}$$

where $\varepsilon(x)$, $h_\lambda^\vee$ and k are given by (4.8), (4.9) and (4.10), respectively. The discrete symmetrized S^s-function expansion is given by

$$f(x) = \sum_{\lambda \in \Lambda_M^s} c_\lambda^s \varphi_\lambda^s(x), \quad \text{where } c_\lambda^s = \frac{1}{kM^3 h_\lambda^\vee} \sum_{x \in F_M^s} \varepsilon(x) f(x) \overline{\varphi_\lambda^s(x)}. \tag{4.20}$$

4.4.1.3. S^s-*functions of* B_3

For a point with coordinates in $\alpha^\vee-$basis (x,y,z) and a weight with coordinates in $\omega-$basis of (a,b,c), the coressponding S^s-functions are explicitly evaluated as

$$
\begin{aligned}
\varphi^s_\lambda(x,y,z) = 2i\,\{&\sin(2\pi(ax+by+cz)) + \sin(2\pi(-ax+(a+b)y+cz))\\
&+\sin(2\pi((a+b)x-by+(2b+c)z)) - \sin(2\pi(ax+(b+c)y-cz))\\
&+\sin(2\pi(bx-(a+b)y+(2a+2b+c)z)) - \sin(2\pi(-ax+(a+b+c)y-cz))\\
&+\sin(2\pi(-(a+b)x+ay+(2b+c)z)) - \sin(2\pi((a+b)x+(b+c)y-(2b+c)z))\\
&-\sin(2\pi((a+b+c)x-(b+c)y+(2b+c)z)) + \sin(2\pi(-bx-ay-(2a+2b+c)z))\\
&-\sin(2\pi(bx+(a+b+c)y-(2a+2b+c)z)) - \sin(2\pi((b+c)x-(a+b+c)y+(2a+2b+c)z))\\
&-\sin(2\pi(-(a+b)x+(a+2b+c)y-(2b+c)z)) - \sin(2\pi((a+2b+c)x-(b+c)y+cz))\\
&-\sin(2\pi(-(a+b+c)x+ay+(2b+c)z)) + \sin(2\pi((a+b+c)x+by-(2b+c)z))\\
&-\sin(2\pi(-bx+(a+2b+c)y-(2a+2b+c)z)) - \sin(2\pi((a+2b+c)x-(a+b+c)y+cz)\\
&-\sin(2\pi(-(b+c)x-ay+(2a+2b+c)z)) + \sin(2\pi((b+c)x+(a+b)y-(2a+2b+c)z)\\
&-\sin(2\pi((b+c)x-(a+2b+c)y+(2a+2b+c)z)) - \sin(2\pi(-(a+2b+c)x+(a+b)y+cz)\\
&+\sin(2\pi((a+2b+c)x-by-cz)) + \sin(2\pi(-(a+b+c)x+(a+2b+c)y-(2b+c)z)\}\,.
\end{aligned}
$$

The coefficients d_λ of continuous orthogonality relations (4.18) are given in Table 4.1.

The discrete grid F^s_M is given by

$$
F^s_M = \left\{ \frac{u^s_1}{M}\omega^\vee_1 + \frac{u^s_2}{M}\omega^\vee_2 + \frac{u^s_3}{M}\omega^\vee_3 \,|\, u^s_0, u^s_1, u^s_2 \in \mathbb{Z}^{\geq 0}, u^s_3 \in \mathbb{N}, u^s_0 + u^s_1 + 2u^s_2 + 2u^s_3 = M \right\} \tag{4.21}
$$

and the corresponding finite set of weights has the form

$$
\Lambda^s_M = \left\{ t^s_1\omega_1 + t^s_2\omega_2 + t^s_3\omega_3 \,|\, t^s_0, t^s_3 \in \mathbb{N}, t^s_1, t^s_2 \in \mathbb{Z}^{\geq 0}, t^s_0 + 2t^s_1 + 2t^s_2 + t^s_3 = M \right\}. \tag{4.22}
$$

$x \in F^s_M$	$\varepsilon(x)$
$[u^s_0, u^s_1, u^s_2, u^s_3]$	48
$[0, u^s_1, u^s_2, u^s_3]$	24
$[u^s_0, 0, u^s_2, u^s_3]$	24
$[u^s_0, u^s_1, 0, u^s_3]$	24
$[0, 0, u^s_2, u^s_3]$	12
$[u^s_0, 0, 0, u^s_3]$	8
$[0, u^s_1, 0, u^s_3]$	8
$[0, 0, 0, u^s_3]$	2

$\lambda \in \Lambda^s_M$	$h^\vee_\lambda$
$[t^s_0, t^s_1, t^s_2, t^s_3]$	1
$[t^s_0, 0, t^s_2, t^s_3]$	2
$[t^s_0, t^s_1, 0, t^s_3]$	2
$[t^s_0, 0, 0, t^s_3]$	6

$x \in F^l_M$	$\varepsilon(x)$
$[u^l_0, u^l_1, u^l_2, u^l_3]$	48
$[u^l_0, u^l_1, u^l_2, 0]$	24

$\lambda \in \Lambda^l_M$	$h^\vee_\lambda$
$[t^l_0, t^l_1, t^l_2, t^l_3]$	1
$[0, t^l_1, t^l_2, t^l_3]$	2
$[t^l_0, t^l_1, t^l_2, 0]$	2
$[0, t^l_1, t^l_2, 0]$	4

TAB. 4.2. The coefficients $\varepsilon(x)$ and $h^\vee_\lambda$ of B_3. All variables u^s_i, t^s_i and u^l_i, t^l_i, $i = 0, 1, 2, 3$, are assumed to be natural numbers.

The number of points in each of these grids are given as

$$
\begin{aligned}
|F^s_{2k}| = |\Lambda^s_{2k}| &= \frac{1}{6}k(k+1)(2k+1) \\
|F^s_{2k+1}| = |\Lambda^s_{2k+1}| &= \frac{1}{3}k(k+1)(k+2).
\end{aligned}
$$

The grid F^s_{10} of B_3 is depicted in Figure 4.3. The coefficients $\varepsilon(x)$ and $h^\vee_\lambda$ of discrete orthogonality relations (4.19) are given in Table 4.2. Each point $x \in F^s_M$ and each weight $\lambda \in \Lambda^s_M$ are represented by the coordinates $[u^s_0,\, u^s_1,\, u^s_2,\, u^s_3]$ and $[t^s_0,\, t^s_1,\, t^s_2,\, t^s_3]$ from defining equations (4.21), (4.22), respectively.

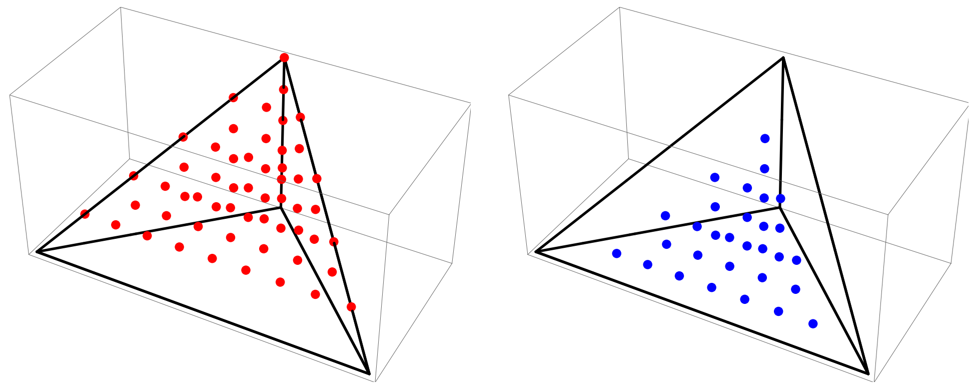

FIG. 4.3. The grids F^s_{10} and F^l_{10} of B_3. On the left–hand side are depicted the points of the grid F^s_{10}, given by (4.21); on the right–hand side are depicted the points of F^l_{10}, given by (4.30).

4.4.1.4. *S^s–functions of C_3*

For a point with coordinates in $\alpha^\vee$–basis (x, y, z) and a weight with coordinates in ω–basis of (a, b, c), the coressponding S^s–functions are explicitly evaluated as

$$
\begin{aligned}
\varphi^s_\lambda(x,y,z) = 2\,\{&\cos(2\pi(ax+by+cz)) - \cos(2\pi(-ax+(a+b)y+cz)) \\
&- \cos(2\pi((a+b)x-by+(b+c)z)) + \cos(2\pi(ax+(b+2c)y-cz)) \\
&+ \cos(2\pi(bx-(a+b)y+(a+b+c)z)) - \cos(2\pi(-ax+(a+b+2c)y-cz)) \\
&+ \cos(2\pi(-(a+b)x+ay+(b+c)z)) - \cos(2\pi((a+b)x+(b+2c)y-(b+c)z)) \\
&- \cos(2\pi((a+b+2c)x-(b+2c)y+(b+c)z)) - \cos(2\pi(-bx-ay-(a+b+c)z)) \\
&+ \cos(2\pi(bx+(a+b+2c)y-(a+b+c)z)) + \cos(2\pi((b+2c)x-(a+b+2c)y+(a+b+c)z)) \\
&+ \cos(2\pi(-(a+b)x+(a+2b+2c)y-(b+c)z)) + \cos(2\pi((a+2b+2c)x-(b+2c)y+cz)) \\
&+ \cos(2\pi(-(a+b+2c)x+ay+(b+c)z)) - \cos(2\pi((a+b+2c)x+by-(b+c)z)) \\
&- \cos(2\pi(-bx+(a+2b+2c)y-(a+b+c)z)) - \cos(2\pi((a+2b+2c)x-(a+b+2c)y+cz) \\
&- \cos(2\pi(-(b+2c)x-ay+(a+b+c)z)) + \cos(2\pi((b+2c)x+(a+b)y-(a+b+c)z) \\
&- \cos(2\pi((b+2c)x-(a+2b+c)y+(a+b+c)z)) - \cos(2\pi(-(a+2b+2c)x+(a+b)y+cz) \\
&+ \cos(2\pi((a+2b+2c)x-by-cz)) + \cos(2\pi(-(a+b+2c)x+(a+2b+2c)y-(b+c)z)\}\,.
\end{aligned}
$$

The coefficients d_λ of continuous orthogonality relations (4.18) are given in Table 4.1.

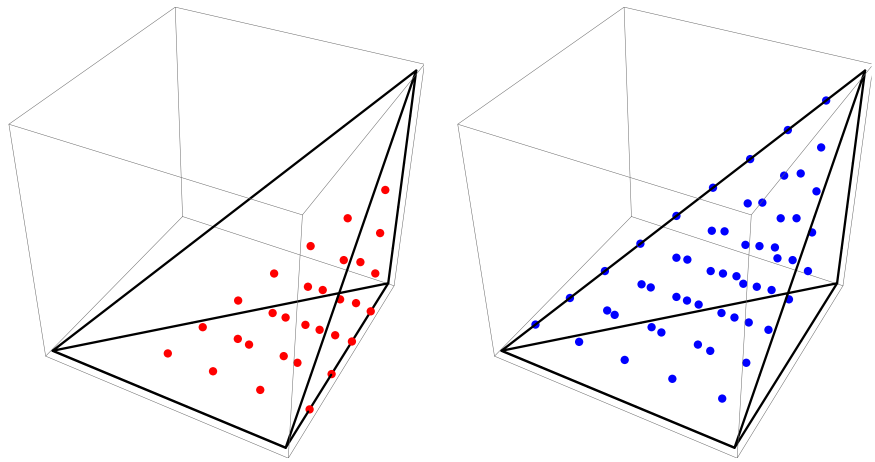

FIG. 4.4. The grids F^s_{10} and F^l_{10} of C_3. On the left–hand side are depicted the points of the grid F^s_{10}, given by (4.23); on the right–hand side are depicted the points of F^l_{10}, given by (4.32).

The discrete grid F^s_M is given by

$$F^s_M = \left\{ \frac{u^s_1}{M}\omega^\vee_1 + \frac{u^s_2}{M}\omega^\vee_2 + \frac{u^s_3}{M}\omega^\vee_3 \,|\, u^s_0, u^s_3 \in \mathbb{Z}^{\geq 0}, u^s_1, u^s_2 \in \mathbb{N}, u^s_0 + 2u^s_1 + 2u^s_2 + u^s_3 = M \right\} \tag{4.23}$$

and the corresponding finite set of weights has the form

$$\Lambda^s_M = \left\{ t^s_1\omega_1 + t^s_2\omega_2 + t^s_3\omega_3 \,|\, t^s_0, t^s_1, t^s_2 \in \mathbb{N}, t^s_0 \in \mathbb{Z}^{\geq 0}, t^s_0 + t^s_1 + 2t^s_2 + 2t^s_3 = M \right\}. \tag{4.24}$$

The number of points in each of these grids are given as

$$|F^s_{2k}| = |\Lambda^s_{2k}| = \frac{1}{6}k(k-1)(2k-1)$$
$$|F^s_{2k+1}| = |\Lambda^s_{2k+1}| = \frac{1}{3}k(k+1)(k-1).$$

The grid F^s_{10} of C_3 is depicted in Figure 4.4. The coefficients $\varepsilon(x)$ and $h^\vee_\lambda$ of discrete orthogonality relations (4.19) are given in Table 4.3. Each point $x \in F^s_M$ and each weight $\lambda \in \Lambda^s_M$ are represented by the coordinates $[u^s_0,\, u^s_1,\, u^s_2,\, u^s_3]$ and $[t^s_0,\, t^s_1,\, t^s_2,\, t^s_3]$ from defining equations (4.23), (4.24), respectively.

$x \in F^s_M$	$\varepsilon(x)$
$[u^s_0, u^s_1, u^s_2, u^s_3]$	48
$[0, u^s_1, u^s_2, u^s_3]$	24
$[u^s_0, u^s_1, u^s_2, 0]$	24
$[0, u^s_1, u^s_2, 0]$	12

$\lambda \in \Lambda^s_M$	$h^\vee_\lambda$
$[t^s_0, t^s_1, t^s_2, t^s_3]$	1
$[t^s_0, t^s_1, t^s_2, 0]$	2

$x \in F^l_M$	$\varepsilon(x)$
$[u^l_0, u^l_1, u^l_2, u^l_3]$	48
$[u^l_0, 0, u^l_1, u^l_2]$	24
$[u^l_0, u^l_1, 0, u^l_3]$	24
$[u^l_0, 0, 0, u^l_3]$	8

$\lambda \in \Lambda^l_M$	$h^\vee_\lambda$
$[t^l_0, t^l_1, t^l_2, t^l_3]$	1
$[0, t^l_1, t^l_2, t^l_3]$	2
$[t^l_0, 0, t^l_2, t^l_3]$	2
$[t^l_0, t^l_1, 0, t^l_3]$	2
$[0, 0, t^l_2, t^l_3]$	4
$[t^l_0, 0, 0, t^l_3]$	6
$[0, t^l_1, 0, t^l_3]$	6
$[0, 0, 0, t^l_3]$	24

TAB. 4.3. The coefficients $\varepsilon(x)$ and $h^\vee_\lambda$ of C_3. All variables u^s_i, t^s_i and u^l_i, t^l_i, $i = 0, 1, 2, 3$, are assumed to be natural numbers.

4.4.1.5. *Example of S^s- functions interpolation*

Consider an arbitrary $M \in \mathbb{N}$ and let f be a function sampled on the grid F^s_M. An $f-$interpolating function $I^s_M : \mathbb{R}^3 \to \mathbb{C}$ is defined as

$$I^s_M(x) = \sum_{\lambda \in \Lambda^s_M} c^s_\lambda \varphi^s_\lambda(x), \tag{4.25}$$

where coefficients $c^s_\lambda \in \mathbb{C}$ are calculated from (4.20). As a specific example of a model function, consider the following smooth characteristic function

$$f_{\alpha,\beta,x_0,y_0,z_0}(x, y, z) = \begin{cases} 1 & \text{if } r < \alpha\,, \\ 0 & \text{if } r > \beta\,, \\ e \exp\left(\left(\dfrac{r-\alpha}{\beta-\alpha}\right)^2 - 1\right)^{-1} & \text{otherwise}\,, \end{cases} \tag{4.26}$$

where $\alpha, \beta, x_0, y_0, z_0 \in \mathbb{R}$ and $r = \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}$. The parameters in (4.26) are set to the values $(\alpha, \beta) = (\frac{1}{20}, \frac{1}{9})$ and $(x_0, y_0, z_0) = (\frac{11}{20}, \frac{1}{3}, \frac{1}{8})$.

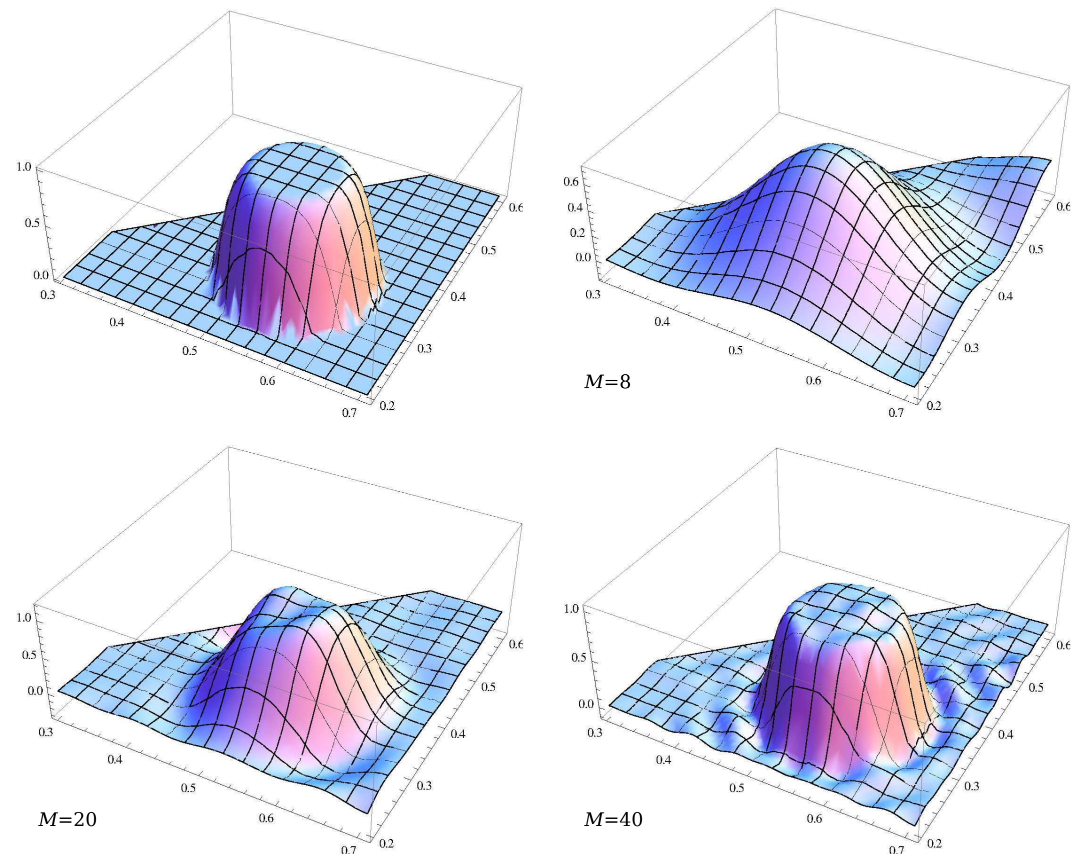

FIG. 4.5. The graph cut ($z = \frac{1}{8}$) of the function f_1 and the graph cuts of the f_1−interpolations I^s_M of C_3 for $M = 8$, 20 and 40.

The resulting function is denoted by f_1 and sampled on the grid F^s_M of C_3. Fixing the third coordinate $z = \frac{1}{8}$, Figure 4.5 shows graph cuts of f_1 and the f_1−interpolating functions I^s_M for $M = 8$, 20 and 40. Integral error estimates of the form $\int_{F^s} |f_1 - I^s_M|^2 \, \mathrm{d}x$ are listed in Table 4.4.

4.4.2. S^l−functions

Choosing $\sigma = \sigma^l$, so–called S^l−functions **[37]** are obtained from (4.14). Following **[17]**, these functions are here denoted by the symbol $\varphi^l_\lambda \equiv \psi^{\sigma^l}_\lambda$. The antiinvariance (4.16) with respect to the long reflections r_α, $\alpha \in \Delta_l$ together with shift invariance (4.15) imply zero values of S^l−functions on the boundary H^l,

$$\varphi^l_\lambda(a') = 0, \quad a' \in H^l.$$

Therefore, the functions φ^l_λ are considered on the fundamental domain $F^l = F \setminus H^l$ only. Similarly, the antiinvariance (4.17) restricts the weights $\lambda \in P$ to the set

P^{+l}. Thus, we have

$$\varphi_\lambda^l(x) = \sum_{w\in W} \sigma^l(w)\, e^{2\pi i\langle w\lambda,\, x\rangle}, \qquad x \in F^l,\ \lambda \in P^{+l}.$$

4.4.2.1. *Continuous orthogonality and S^l-transforms*

For any two weights $\lambda, \lambda' \in P^{+l}$ the corresponding S^l-functions are orthogonal on F^l

$$\int_{F^l} \varphi_\lambda^l(x)\overline{\varphi_{\lambda'}^l(x)}\,\mathrm{d}x = K\, d_\lambda\, \delta_{\lambda\lambda'} \tag{4.27}$$

where d_λ, K are given by (4.6), (4.7), respectively. The S^l-functions determine symmetrized Fourier series expansions,

$$f(x) = \sum_{\lambda\in P^{+l}} c_\lambda^l \varphi_\lambda^l(x), \quad \text{where } c_\lambda^l = \frac{1}{K d_\lambda}\int_{F^l} f(x)\overline{\varphi_\lambda^l(x)}\,\mathrm{d}x.$$

4.4.2.2. *Discrete orthogonality and discrete S^l-transforms*

The finite set of points is given by

$$F_M^l = \frac{1}{M}P^\vee/Q^\vee \cap F^l$$

and the corresponding finite set of weights as

$$\Lambda_M^l = P/MQ \cap MF^{l\vee}.$$

Then, for $\lambda, \lambda' \in \Lambda_M^l$, the following discrete orthogonality relations hold,

$$\sum_{x\in F_M^l} \varepsilon(x)\varphi_\lambda^l(x)\overline{\varphi_{\lambda'}^l(x)} = kM^3 h_\lambda^\vee \delta_{\lambda\lambda'} \tag{4.28}$$

where $\varepsilon(x)$, $h_\lambda^\vee$ and k are given by (4.8), (4.9) and (4.10), respectively. The discrete symmetrized S^l-function expansion is given by

$$f(x) = \sum_{\lambda\in\Lambda_M^l} c_\lambda^l \varphi_\lambda^l(x), \quad \text{where } c_\lambda^l = \frac{1}{kM^3 h_\lambda^\vee}\sum_{x\in F_M^l} \varepsilon(x) f(x)\overline{\varphi_\lambda^l(x)}. \tag{4.29}$$

4.4.2.3. S^l-*functions of* B_3

For a point with coordinates in $\alpha^\vee-$basis (x,y,z) and a weight with coordinates in $\omega-$basis of (a,b,c), the coressponding S^l-functions are explicitly evaluated as

$$
\begin{aligned}
\varphi^l_\lambda(x,y,z) &= 2\left\{\cos(2\pi(ax+by+cz)) - \cos(2\pi(-ax+(a+b)y+cz))\right.\\
&- \cos(2\pi((a+b)x-by+(2b+c)z)) + \cos(2\pi(ax+(b+c)y-cz))\\
&+ \cos(2\pi(bx-(a+b)y+(2a+2b+c)z)) - \cos(2\pi(-ax+(a+b+c)y-cz))\\
&+ \cos(2\pi(-(a+b)x+ay+(2b+c)z)) - \cos(2\pi((a+b)x+(b+c)y-(2b+c)z))\\
&- \cos(2\pi((a+b+c)x-(b+c)y+(2b+c)z)) - \cos(2\pi(-bx-ay-(2a+2b+c)z))\\
&+ \cos(2\pi(bx+(a+b+c)y-(2a+2b+c)z)) + \cos(2\pi((b+c)x-(a+b+c)y+(2a+2b+c)z))\\
&+ \cos(2\pi(-(a+b)x+(a+2b+c)y-(2b+c)z)) + \cos(2\pi((a+2b+c)x-(b+c)y+cz))\\
&+ \cos(2\pi(-(a+b+c)x+ay+(2b+c)z)) - \cos(2\pi((a+b+c)x+by-(2b+c)z))\\
&- \cos(2\pi(-bx+(a+2b+c)y-(2a+2b+c)z)) - \cos(2\pi((a+2b+c)x-(a+b+c)y+cz)\\
&- \cos(2\pi(-(b+c)x-ay+(2a+2b+c)z)) + \cos(2\pi((b+c)x+(a+b)y-(2a+2b+c)z)\\
&- \cos(2\pi((b+c)x-(a+2b+c)y+(2a+2b+c)z)) - \cos(2\pi(-(a+2b+c)x+(a+b)y+cz)\\
&\left.+ \cos(2\pi((a+2b+c)x-by-cz)) + \cos(2\pi(-(a+b+c)x+(a+2b+c)y-(2b+c)z)\right\}.
\end{aligned}
$$

The coefficients d_λ of continuous orthogonality relations (4.27) are given in Table 4.1. The discrete grid F^l_M is given by

$$
F^l_M = \left\{\frac{u^l_1}{M}\omega^\vee_1 + \frac{u^l_2}{M}\omega^\vee_2 + \frac{u^l_3}{M}\omega^\vee_3 \,|\, u^l_0, u^l_1, u^l_2 \in \mathbb{N}, u^l_3 \in \mathbb{Z}^{\geq 0}, u^l_0 + u^l_1 + 2u^l_2 + 2u^l_3 = M\right\} \tag{4.30}
$$

and the corresponding finite set of weights has the form

$$
\Lambda^l_M = \left\{t^l_1\omega_1 + t^l_2\omega_2 + t^l_3\omega_3 \,|\, t^l_0, t^l_3 \in \mathbb{Z}^{\geq 0}, t^l_1, t^l_2 \in \mathbb{N}, t^l_0 + 2t^l_1 + 2t^l_2 + t^l_3 = M\right\}. \tag{4.31}
$$

The number of points in each of these grids are given as

$$
\begin{aligned}
|F^l_{2k}| = |\Lambda^l_{2k}| &= \frac{1}{6}k(k-1)(2k-1)\\
|F^l_{2k+1}| = |\Lambda^l_{2k+1}| &= \frac{1}{3}k(k+1)(k-1).
\end{aligned}
$$

The grid F^l_{10} of B_3 is depicted in Figure 4.3. The coefficients $\varepsilon(x)$ and $h^\vee_\lambda$ of discrete orthogonality relations (4.28) are given in Table 4.2. Each point $x \in F^l_M$ and each weight $\lambda \in \Lambda^l_M$ are represented by the coordinates $[u^l_0,\, u^l_1,\, u^l_2,\, u^l_3]$ and $[t^l_0,\, t^l_1,\, t^l_2,\, t^l_3]$ from defining equations (4.30), (4.31), respectively.

4.4.2.4. *S^l–functions of C_3*

For a point with coordinates in $\alpha^\vee$–basis (x, y, z) and a weight with coordinates in ω–basis of (a, b, c), the coressponding S^l–functions are explicitly evaluated as

$$
\begin{aligned}
\varphi^l_\lambda(x,y,z) = 2i\,\{&\sin(2\pi(ax+by+cz)) + \sin(2\pi(-ax+(a+b)y+cz))\\
&+\sin(2\pi((a+b)x-by+(b+c)z)) - \sin(2\pi(ax+(b+2c)y-cz))\\
&+\sin(2\pi(bx-(a+b)y+(a+b+c)z)) - \sin(2\pi(-ax+(a+b+2c)y-cz))\\
&+\sin(2\pi(-(a+b)x+ay+(b+c)z)) - \sin(2\pi((a+b)x+(b+2c)y-(b+c)z))\\
&-\sin(2\pi((a+b+2c)x-(b+2c)y+(b+c)z)) + \sin(2\pi(-bx-ay-(a+b+c)z))\\
&-\sin(2\pi(bx+(a+b+2c)y-(a+b+c)z)) - \sin(2\pi((b+2c)x-(a+b+2c)y+(a+b+c)z))\\
&-\sin(2\pi(-(a+b)x+(a+2b+2c)y-(b+c)z)) - \sin(2\pi((a+2b+2c)x-(b+2c)y+cz))\\
&-\sin(2\pi(-(a+b+2c)x+ay+(b+c)z)) + \sin(2\pi((a+b+2c)x+by-(b+c)z))\\
&-\sin(2\pi(-bx+(a+2b+2c)y-(a+b+c)z)) - \sin(2\pi((a+2b+2c)x-(a+b+2c)y+cz)\\
&-\sin(2\pi(-(b+2c)x-ay+(a+b+c)z)) + \sin(2\pi((b+2c)x+(a+b)y-(a+b+c)z)\\
&-\sin(2\pi((b+2c)x-(a+2b+c)y+(a+b+c)z)) - \sin(2\pi(-(a+2b+2c)x+(a+b)y+cz)\\
&+\sin(2\pi((a+2b+2c)x-by-cz)) + \sin(2\pi(-(a+b+2c)x+(a+2b+2c)y-(b+c)z)\}\,.
\end{aligned}
$$

The coefficients d_λ of continuous orthogonality relations (4.27) are given in Table 4.1. The discrete grid F^l_M is given by

$$
F^l_M = \left\{ \frac{u^l_1}{M}\omega^\vee_1 + \frac{u^l_2}{M}\omega^\vee_2 + \frac{u^l_3}{M}\omega^\vee_3 \,|\, u^l_0, u^l_3 \in \mathbb{N}, u^l_1, u^l_2 \in \mathbb{Z}^{\geq 0}, u^l_0 + 2u^l_1 + 2u^l_2 + u^l_3 = M \right\} \tag{4.32}
$$

and the corresponding finite set of weights has the form

$$
\Lambda^l_M = \left\{ t^l_1\omega_1 + t^l_2\omega_2 + t^l_3\omega_3 \,|\, t^l_0, t^l_1, t^l_2 \in \mathbb{Z}^{\geq 0}, t^l_3 \in \mathbb{N}, t^l_0 + t^l_1 + 2t^l_2 + 2t^l_3 = M \right\}. \tag{4.33}
$$

M	$\int_{F^s} \lvert f_1 - I^s_M\rvert^2 \,\mathrm{d}x$	$\int_{F^l} \lvert f_2 - I^l_M\rvert^2 \,\mathrm{d}x$
8	$2162,5 \times 10^{-6}$	$574,87 \times 10^{-6}$
16	$350,62 \times 10^{-6}$	$202,74 \times 10^{-6}$
24	$77,45 \times 10^{-6}$	$57,16 \times 10^{-6}$
32	$32,14 \times 10^{-6}$	$13,07 \times 10^{-6}$
40	$15,88 \times 10^{-6}$	$12,73 \times 10^{-6}$

TAB. 4.4. Integral error estimates of the interpolations I^s_M ans I^l_M.

The number of points in each of these grids are given as

$$
\begin{aligned}
|F^l_{2k}| = |\Lambda^l_{2k}| &= \frac{1}{6}k(k+1)(2k+1)\\
|F^l_{2k+1}| = |\Lambda^l_{2k+1}| &= \frac{1}{3}k(k+1)(k+2).
\end{aligned}
$$

The grid F^l_{10} of C_3 is depicted in Figure 4.4. The coefficients $\varepsilon(x)$ and $h^\vee_\lambda$ of discrete orthogonality relations (4.28) are given in Table 4.3. Each point $x \in F^l_M$ and each weight $\lambda \in \Lambda^l_M$ are represented by the coordinates $[u^l_0,\, u^l_1,\, u^l_2,\, u^l_3]$ and $[t^l_0,\, t^l_1,\, t^l_2,\, t^l_3]$ from defining equations (4.32), (4.33), respectively.

4.4.2.5. *Example of S^l- functions interpolation*

Consider an arbitrary $M \in \mathbb{N}$ and let f be a function sampled on the grid F^l_M. An $f-$interpolating function $I^l_M : \mathbb{R}^3 \to \mathbb{C}$ is defined as

$$
I^l_M(x) = \sum_{\lambda\in\Lambda^l_M} c^l_\lambda \varphi^l_\lambda(x), \tag{4.34}
$$

where coefficients $c^l_\lambda \in \mathbb{C}$ are calculated from (4.29). The parameters in (4.26) are set to the values $(\alpha,\beta) = (\frac{1}{20}, \frac{1}{9})$ and $(x_0, y_0, z_0) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{8})$ and the resulting function, denoted by f_2, is sampled on the grid F^l_M of B_3. Fixing the third coordinate $z = \frac{1}{8}$, Figure 4.6 shows graph cuts of f_2 and the f_2-interpolating functions I^l_M for $M = 8$, 20 and 40. Integral error estimates of the form $\int_{F^l} |f_2 - I^l_M|^2 \,\mathrm{d}x$ are listed in Table 4.4.

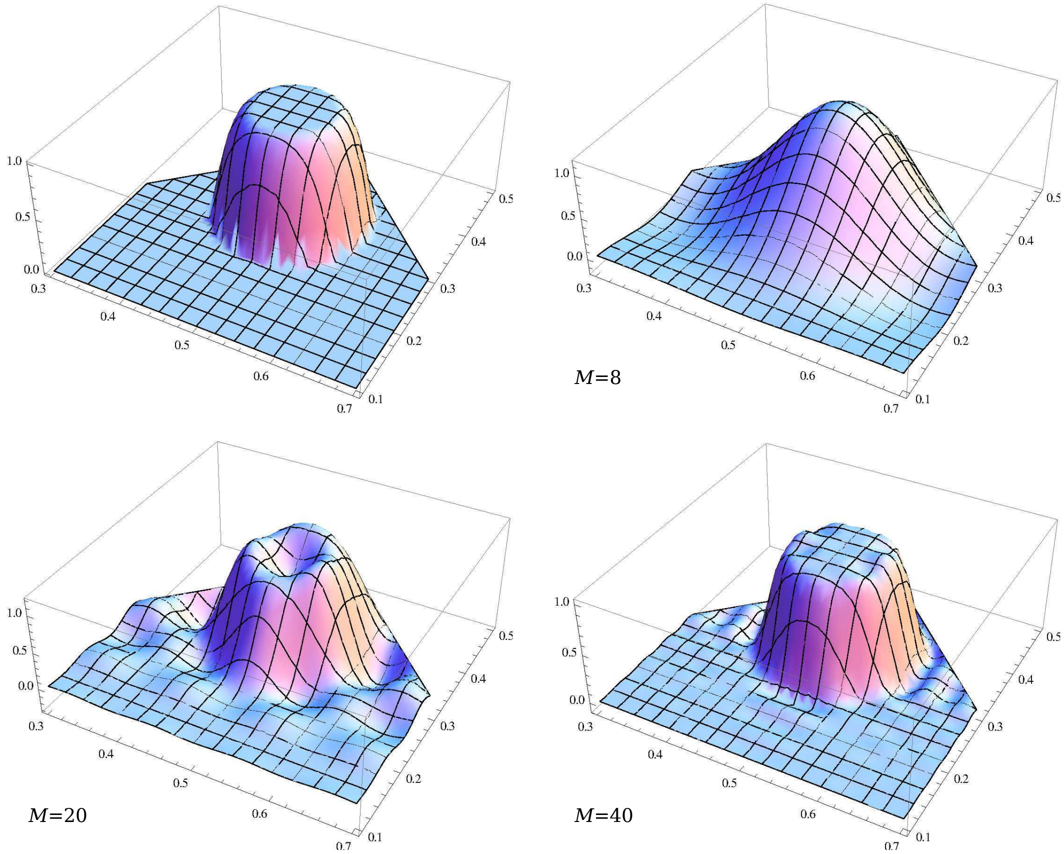

FIG. 4.6. The graph cut ($z = \frac{1}{8}$) of the function f_2 and the graph cuts of the f_2-interpolations I^l_M of B_3 for $M = 8$, 20 and 40.

4.5. Concluding Remarks

- $C-$ and $S-$functions are related by the well known Weyl character formula [**20**] for irreducible representations of compact semisimple Lie groups. Let χ_λ be the character of an irreducible representation of the group G with the highest root λ. Then we have

$$\chi_\lambda(x) = \sum_{\mu \in P_+} m^\lambda_\mu \Phi_\mu(x) = \frac{\varphi_{\lambda+\rho}(x)}{\varphi_\rho(x)}\,,$$

where Φ_λ and φ_λ are the orbit functions related to the Weyl group W of G, ρ is the half sum of positive roots in the root system of W. The numbers m^λ_μ are the multiplicities of the dominant weights μ in the weight system of the irreducible representations with the highest weight λ. Extensive tables of the multiplicities for simple Lie groups up to rank 12 can be found in [**48**].

- The product of two S^s-functions or two S^l-functions of B_3 or of C_3, with the same arguments $x \in \mathbb{R}^3$ but with different weights λ and λ', decomposes into the sum of $C-$functions:

$$\varphi^s_\lambda(x) \cdot \varphi^s_{\lambda'}(x) = \sum_{w \in W} \sigma^s(w) \Phi_{\lambda + w\lambda'}(x),$$
$$\varphi^l_\lambda(x) \cdot \varphi^l_{\lambda'}(x) = \sum_{w \in W} \sigma^l(w) \Phi_{\lambda + w\lambda'}(x).$$

- The two examples of interpolation show promising behaviour of the $f-$interpolating functions I^s_M and I^l_M. Visual inspection as well as the integral error estimates of the interpolations of the model functions f_1 and f_2 lead to the qualitative conclusion that the interpolation error is rather small once the minimal distances of the lattice grids become smaller than the diameters β of the $3D$ jumps of the model functions. The existence of general conditions for the convergence of the functional series $\{I^s_M\}_{M=1}^\infty$, $\{I^l_M\}_{M=1}^\infty$ deserves further study.
- The Gaussian cubature formula, built on Chebyshev polynomials of the second kind (equivalently, the characters of A_1), is known to lead to optimal interpolation of polynomial functions of one variable [**59**]. An analogous result extended to n-variable functions with underlying A_n-symmetry is known [**33**]. Recently it has been further extended to simple Lie groups of all types [**40**]. A curious question, which has not yet been answered, is about possibilities to build cubature formulas using the hybrid characters of B_3, C_3 and other higher rank simple Lie groups.

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